

Uniform Asymptotics for Compound Poisson Processes with Regularly Varying Jumps and Vanishing Drift

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Abstract

This paper addresses heavy-tailed asymptotics of functionals of a class of spectrally one-sided Lévy process that remain valid in a near-critical regime. This complements recent similar results that have been obtained for the all-time supremum of such processes. Specifically, we consider local asymptotics of the all-time supremum, the supremum of the process until exiting $[0, \infty)$, the maximum jump until that time, and the time it takes until exiting $[0, \infty)$. The proofs rely, among other things, on properties of scale functions.

Keywords: compound Poisson process, $M/G/1$ queue, heavy traffic, large deviations, uniform asymptotics, first passage time, supremum

1 Introduction

The analysis of spectrally one-sided Lévy processes is a topic of fundamental interest in the stochastic processes literature [26] and arises in many applications, such as queueing [13] and insurance risk theory [3, 17]. More generally, Lévy processes and various functionals have been studied extensively over the last decades through fluctuation theory, leading to many interesting and useful, both exact and asymptotic, results. Analyses of asymptotic regimes have significantly contributed to the present understanding of the qualitative behaviour of Lévy processes. In some cases these asymptotic regimes lead to different approximations and therefore do not commute, indicating a phase transition. It is of interest to investigate where the phase transition occurs and to determine how robust each resulting approximation is.

Phase transitions and robustness form a broad motivation for the current study, which considers two asymptotic regimes that are essentially the central limit regime and the large deviations regime. We consider compound Poisson processes with deterministic drift of the form

$$X^\rho(t) := X_0 + \sum_{i=1}^{N^\rho(t)} B_i - t, \quad (1.1)$$

where $N^\rho(t), t \geq 0$, is a Poisson process with a rate that depends on a drift parameter ρ . With a slight abuse of terminology, we call X^ρ a compound Poisson process throughout this paper, and investigate the asymptotic behaviour of various functionals of X^ρ under the assumption that the jump sizes B_i have a regularly varying tail with finite variance. The initial condition X_0 is equal in distribution to B_i and independent of ρ . The long-term drift $E[X^\rho(1) - X_0]$ of the process is negative, and of order $1 - \rho$. In the central limit regime, we let $\rho \uparrow 1$ so that the long-term drift tends to zero.

A functional that has received ample attention in the literature is the all-time supremum $M_\infty^\rho := \sup_{t \geq 0} X^\rho(t)$. Specifically, the behaviour of $\mathbb{P}(M_\infty^\rho > x)$ has been studied in both the central limit and the large deviation regimes. On the one hand, the central limit regime considers the behaviour of M_∞^ρ as ρ tends to one, yielding so-called heavy traffic or diffusion approximations. In this regime, the distribution of $(1 - \rho)M_\infty^\rho$ behaves like an exponential distribution under any jump size distribution with finite second moment [34, 35]. On the other hand, the large deviation regime regards the behaviour of $\mathbb{P}(M_\infty^\rho > x)$ for fixed ρ while x tends to infinity and provides heavy-tail approximations. Embrechts and Veraverbeke [17] imply within the setting of this paper that the tail distribution $\mathbb{P}(M_\infty^\rho > x)$ asymptotically behaves like the tail of the integrated tail distribution $\int_x^\infty \mathbb{P}(B_1 > t) dt / \mathbb{E}[B_1]$ of B_1 ; see also Klüppelberg et al. [22], Maulik and Zwart [28] and Foss et al. [18]. This behaviour holds up to a constant which in itself scales as the reciprocal of the long-term drift $(1 - \rho)^{-1}$ when $\rho \uparrow 1$. These approximations demonstrate a fundamental contrast between the two regimes, as the distribution of $(1 - \rho)M_\infty^\rho$ can exhibit a heavy or exponential tail depending on the regime under consideration.

Motivated by the contrast between these two regimes, Olvera-Cravioto et al. [30] establish an explicit threshold

$$\tilde{x}_\rho := \mu(\alpha - 2) \frac{1}{1 - \rho} \log \frac{1}{1 - \rho}, \quad (1.2)$$

for some $\alpha > 2$ and $\mu > 0$ as specified in the next section, where the two regimes connect. That is, they show that the heavy-tail approximation is valid for $x > \tilde{x}_\rho$ and the diffusion approximation holds for $x < \tilde{x}_\rho$ as $\rho \uparrow 1$. Moreover, Olvera-Cravioto et al. [30] prove that the asymptotic tail behaviour $\mathbb{P}(M_\infty^\rho > x)$ is uniformly characterized by the heavy-tail and diffusion approximations. Olvera-Cravioto et al. [30] consider regularly varying jump size distributions, yet this restriction is loosened to a more general subclass of subexponential distributions in Olvera-Cravioto and Glynn [31]. They show that, in contrast to the regularly varying case, a third regime may be required in order to describe the behaviour of the all-time supremum if the jump size distribution has a sufficiently light tail. Blanchet and Lam [7] present a refinement of the transition from the large deviations to the central limit regime. Extensions of the aforementioned results to more general random walks are presented in Kugler and Wachtel [24] and Denisov and Kugler [15].

All of the above mentioned works focus on global asymptotics of the all-time supremum functional M_∞^ρ , and one may wonder how robust the obtained insights are. For example, another well-studied functional of Lévy processes is the first passage time of zero, τ^ρ , which among others may characterize a busy period duration in queueing theory. Ideally, the asymptotic behaviour of the passage time could be characterized in a similar way as the all-time supremum. However, there does not seem to exist a weak limit of this passage time as $\rho \uparrow 1$ when X_0 is independent of ρ , and as such there can be no smooth transition from the central limit to the large deviation regime. The same argument holds for the supremum functional $M_\tau^\rho := \sup_{t < \tau^\rho} X^\rho(t)$. In contrast, a series of prior works [4, 29, 36] obtain useful asymptotic approximations for τ^ρ in the large deviation regime under subexponential jump sizes, while M_τ^ρ has been considered for fixed ρ in Asmussen [1]. The latter result is actually applied in the sample-path analysis by Zwart [36] to obtain the tail behaviour of τ^ρ . A similar construction is used in this paper.

Motivated by these considerations, we consider $\mathbb{P}(\tau^\rho > x)$ as $x \rightarrow \infty$ while the long-term drift is vanishing. It turns out that our threshold, for which the large deviations approximation can be shown to be valid, is much larger than threshold (1.2). In particular, we introduce the threshold

$$x_\rho^* := \frac{1}{(1 - \rho)^2} \left(\log \frac{1}{1 - \rho} \right)^{k\alpha}, \quad (1.3)$$

where $k > 1$, so that for all $x \geq x_\rho^*$ the tail probability $\mathbb{P}(\tau^\rho > x)$ asymptotically behaves like $\frac{1}{1 - \rho} \mathbb{P}(B > (1 - \rho)x)$ as $\rho \uparrow 1$. Note that x_ρ^* is slightly larger than $\tilde{x}_\rho / (1 - \rho)$. The latter threshold would be essential: we show that the asymptotic behaviour of $\mathbb{P}(\tau^\rho > x)$ coincides

with $\mathbb{P}(M_\tau^\rho > (1 - \rho)x)$. Intuitively, if the process hits zero after time x , then it is likely that the process obeyed the long-term drift after reaching level $(1 - \rho)x$ early in time. Uniform heavy-tail approximations for M_τ^ρ , which are also established in this paper as by-product of independent interest, yield the given asymptotic. The gap between x_ρ^* and $\tilde{x}_\rho/(1 - \rho)$ is required for technical reasons.

Additional theorems that lead to our main result provide uniform heavy-tail approximations on the “local” tail probability $\mathbb{P}(M_\infty^\rho \in [x, x + T])$ of the all-time supremum functional M_∞^ρ ; and the largest jump B_τ^ρ until time τ^ρ . In addition, we derive asymptotic expressions for the conditional expected time of reaching a high level a , given level a is reached before time τ^ρ . The local asymptotics of M_∞^ρ provide a generalization of Corollary 2.1(b) in Olvera-Cravioto et al. [30] and are obtained in a similar fashion via a decomposition of the so-called Pollaczek-Khinchine formula. Our theorem on passage times relies heavily on fluctuation theory for Levy processes; specifically, it relies on the theory of scale functions. A recent review article on and examples of scale functions can be found in Kuznetsov et al. [25] and Hubalek and Kyprianou [21], respectively.

The paper is structured as follows. A precise description of the model and an introduction to the notation used can be found in Section 2. Section 3 presents and discusses our results; in particular, Theorems 1 and 5 display our main results. The five subsequent sections are each devoted to the proof of one theorem.

2 Preliminaries

Let $\{B\} \cup \{B_i\}_{i=0}^\infty$ be a sequence of independent and identically distributed (i.i.d.) regularly varying random variables [cf. 6] with mean $\mathbb{E}[B] > 0$ and finite variance σ_B^2 . More specifically, the common distribution function $F_B : \mathbb{R} \rightarrow [0, 1]$, $F(0) = 0$ of the B_i is characterized by its tail

$$\overline{F}_B(x) := \mathbb{P}(B > x) = L(x)x^{-\alpha}, \quad (2.1)$$

where $\alpha > 2$, $\alpha \neq 3$ and $L(x)$ is a slowly varying function: $\lim_{x \rightarrow \infty} L(ax)/L(x) = 1$ for all $a > 0$. A key property of such distributions is that $\mathbb{E}[B^p] < \infty$ for $p < \alpha$ and $\mathbb{E}[B^p] = \infty$ for $p > \alpha$. The α -th moment can be either finite or infinite. For technical reasons, this article does not address the $\alpha = 3$ case. It should be noted that regularly varying distributions are a subclass of subexponential distributions [19], and as such satisfy $\lim_{x \rightarrow \infty} \mathbb{P}(B_1 + \dots + B_n > x)/\mathbb{P}(B_1 > x) = n$.

Define the Poisson process $N(t)$, $t \geq 0$, with rate $1/\mathbb{E}[B]$ and independent of B . Then $N^\rho(t) := N(\rho t)$, $t \geq 0$, is a Poisson process with rate $\rho/\mathbb{E}[B]$ and the process $X^\rho : [0, \infty) \rightarrow \mathbb{R}$ given by

$$X^\rho(t) := X_0 + \sum_{i=1}^{N^\rho(t)} B_i - t \quad (2.2)$$

is a compound Poisson process with initial value $X^\rho(0) = X_0 := B_0$ and long-term drift $\mathbb{E}[X^\rho(1) - X^\rho(0)] = -(1 - \rho) < 0$. The process $X^\rho(t)$ experiences a deterministic decrease of $-t$ and has jumps of size B_i . For this reason we refer to F_B as the jump size distribution. One may also consider $X^\rho(t)$ as a Poisson process that experiences its first jump at time $t = 0$. The total number of jumps then equals $\tilde{N}^\rho(t) := N^\rho(t) + 1$. A challenging asymptotic is provided by $\rho \uparrow 1$, in which case the long-term drift of the process X^ρ tends to zero.

The first time that $X^\rho(t)$ exceeds level x is denoted by $\sigma^\rho(x) := \inf\{t \geq 0 : X^\rho(t) \geq x\}$, whereas the first hitting time of level zero is indicated by $\tau^\rho := \inf\{t \geq 0 : X^\rho(t) = 0\}$. Note the difference between $\sigma^\rho(0)$ ($= 0$ almost surely), the first time that X^ρ is positive; and τ^ρ (> 0 almost surely), the first time that X^ρ is non-positive. Of primary interest in this article are the supremum M_τ^ρ of $X^\rho(t)$ until the first down-crossing of level zero, i.e. $M_\tau^\rho := \sup\{X^\rho(t) :$

$0 \leq t \leq \tau^\rho$, and the all-time supremum $M_\infty^\rho := \sup\{X^\rho(t) : t \geq 0\}$ of the Lévy process. We also derive a result on the number of jumps before time τ^ρ , $N^\rho(\tau^\rho)$, and on the largest jump B_τ^ρ before time τ^ρ : $B_\tau^\rho := \sup\{B_i : 0 \leq i \leq N^\rho(\tau^\rho)\}$.

Consider the sequence of i.i.d. random variables $\{B^*\} \cup \{B_i^*\}_{i=1}^\infty$ with distribution function F_{B^*} . F_{B^*} is the integrated tail distribution of B and will be referred to as excess jump size distribution. The excess jump size distribution can be characterized by its density $f_{B^*}(x) = \frac{1}{\mathbb{E}[B]} \mathbb{P}(B > x)$ and has finite mean $\mu = \mathbb{E}[B^2]/(2\mathbb{E}[B]) < \infty$. It is assumed that B^* and B_i^* are independent of N^ρ, B, B_i and for all relevant indices. Since B is regularly varying, Karamata's Theorem [6, Theorem 1.5.11] claims that the tail distribution of B^* ,

$$\overline{F}_{B^*}(x) = \frac{1}{\mathbb{E}[B]} \int_x^\infty \mathbb{P}(B > t) dt \sim \frac{1}{(\alpha - 1)\mathbb{E}[B]} L(x)x^{-\alpha+1},$$

is also regularly varying, where $f(z) \sim g(z)$ if and only if $\lim_{z \uparrow z^*} f(z)/g(z) = 1$ for some limiting value $z^* \in \{1, \infty\}$. In this paper, the limit of interest is either $\rho \uparrow 1, x \rightarrow \infty$ or $a \rightarrow \infty$. The proper limit should be clear from the context. Similarly, $f(z) \gtrsim (\lesssim) g(z)$ denotes the relation $\lim_{z \uparrow z^*} f(z)/g(z) \geq (\leq) 1$. We adopt the common conventions $f(z) = \mathcal{O}(g(z))$ if and only if $\lim_{z \uparrow z^*} f(z)/g(z) < \infty$ and $f(z) = o(g(z))$ if and only if $\lim_{z \uparrow z^*} f(z)/g(z) = 0$.

Let $T \in (0, \infty)$ be any positive constant and define the interval $\Delta = [0, T)$. In the remainder of this article we will denote the “local” probability $\mathbb{P}(B^* \in [x, x + T))$ by $F_{B^*}(x + \Delta)$. Furthermore, we adopt the well-known conventions $\lfloor x \rfloor := \max\{n \in \mathbb{N} : n \leq x\}$ and $\lceil x \rceil := \min\{n \in \mathbb{N} : n \geq x\}$.

Many expressions in this article involve constants which do not provide additional insight, and which do not contribute to the global behaviour of the expressions. For this reason, many constants have been replaced by C : a constant whose value may change from line to line.

Most variables that have been introduced so far depend on the parameter ρ . Now that their dependence has been noted, we drop the sub- and superscripts ρ for the remainder of this article. Variables that are introduced in later sections and that depend on ρ will have a sub- or superscript unless mentioned otherwise.

3 Results and discussion

The purpose of this section is to present and discuss our large deviations results in the asymptotic regime where the long-term drift vanishes. Theorem 1 is our first main contribution and illustrates an exact uniform asymptotic for the relation between the all-time supremum M_∞ of the Lévy process $X(t)$ and excess jump size B^* . Theorems 2 and 3 display exact uniform asymptotics for M_τ in terms of the jump size distribution, and in terms of the largest jump before time τ . Theorem 5 is our second main contribution and shows an exact uniform asymptotic for the connection between the first hitting time τ and M_τ . A queuing theoretic interpretation of our results is provided at the end of this section. The proofs of the theorems are deferred to later sections.

Theorem 1. *Suppose $\mathbb{P}(B > x) = L(x)x^{-\alpha}$ for some $\alpha > 2, \alpha \neq 3$ and $L(x)$ slowly varying. Let $\overline{F}_{B^*}(x) = \mathbb{P}(B^* > x) = \frac{1}{\mathbb{E}[B]} \int_x^\infty \mathbb{P}(B > x) dx$ be the integrated tail distribution with mean $\mu = \mathbb{E}[B^2]/(2\mathbb{E}[B])$. Define $x_\rho := k\mu(\alpha - 1)\frac{1}{1-\rho} \log \frac{1}{1-\rho}$ for any $k > 1$. Then*

$$\sup_{x \geq x_\rho} \left| \frac{\mathbb{P}(M_\infty \in x + \Delta)}{\frac{\rho}{1-\rho} \mathbb{P}(B^* \in x + \Delta)} - 1 \right| \rightarrow 0 \quad (3.1)$$

as $\rho \uparrow 1$. Furthermore, (3.1) remains valid for $k = 1$ provided that $L(x)/(\log x)^\alpha \rightarrow \infty$.

Theorem 1 extends Corollary 2.3(b) of Olvera-Cravioto et al. [30], who considered the “global” tail probability $\Delta = [0, \infty)$. Their results are presented in a queueing context, where Theorem 1 states a relation between the steady state waiting time $W \stackrel{d}{=} M_\infty$ in an M/G/1 queue, and the excess jump size distribution. The similarity of the results is also reflected in the proof of the theorem, which greatly depends on the Pollaczek-Khintchine formula and the power law nature of the jump size distribution. A key difference between the proofs is Olvera-Cravioto et al.’s application of the “global” big jump asymptotics as reported by Borovkov and Borovkov [8] versus our usage of the “local” analogues as derived by Denisov et al. [14]. The transition point \tilde{x}_ρ in Olvera-Cravioto et al. [30] differs from x_ρ by a factor $\frac{\alpha-1}{\alpha-2}$, which is an artefact of our analysis of the local tail probability (index α) as opposed to their analysis of the global tail probability (index $\alpha - 1$). Similarly, their $k = 1$ case requires $L(x)$ to asymptotically dominate $(\log x)^{\alpha-1}$ instead of $(\log x)^\alpha$.

A related asymptotic is presented in Theorem 2, where M_∞ and B^* in Theorem 1 have been replaced by M_τ and B , respectively.

Theorem 2. *Suppose that all conditions in Theorem 1 hold. Then*

$$\sup_{x \geq x_\rho} \left| \frac{\mathbb{P}(M_\tau > x)}{\frac{\rho}{1-\rho} \mathbb{P}(B > x)} - 1 \right| \rightarrow 0 \quad (3.2)$$

as $\rho \uparrow 1$. Furthermore, (3.2) remains valid for $k = 1$ provided that $L(x)/(\log x)^\alpha \rightarrow \infty$.

Theorem 2 specifies the asymptotic equivalence presented in Asmussen [2, Theorem 2.1]. As the mean number of jumps until time τ is given by $\mathbb{E}[\tau] = 1/(1 - \rho)$, the above theorem states that the probability of a large supremum M_τ is asymptotically equivalent to the probability of one large jump, multiplied by the expected number of jumps. A key property of regularly varying distributions is that this multiplication is asymptotically equivalent to the largest jump before time τ , B_τ . This is made explicit by Theorem 3.

Theorem 3. *Suppose that all conditions in Theorem 1 hold. Then*

$$\sup_{x \geq x_\rho} \left| \frac{\mathbb{P}(M_\tau > x)}{\mathbb{P}(B_\tau > x)} - 1 \right| \rightarrow 0 \quad (3.3)$$

as $\rho \uparrow 1$. Furthermore, (3.3) remains valid for $k = 1$ provided that $L(x)/(\log x)^\alpha \rightarrow \infty$.

Theorems 2 and 3 both consider M_τ . In contrast, Theorem 4 regards the time $\sigma(a)$ at which $X(t)$ exceeds some high level a . It states that if the level under consideration is sufficiently large, then this level is achieved relatively early in time; specifically, it states that if a is sufficiently large then this level is attained in time $\mathcal{O}(1/(1 - \rho)^2)$.

Theorem 4. *Suppose $\mathbb{P}(B > x) = L(x)x^{-\alpha}$ for some $\alpha > 2, \alpha \neq 3$ and $L(x)$ slowly varying. Define $a_\rho^* := 2k\mu(\alpha - 1)\frac{1}{1-\rho} \log \frac{1}{1-\rho}$ for some $k > 1$. Then for any $y > 0$,*

$$\sup_{a \geq a_\rho^*} \mathbb{E}[\sigma(a) \mid \sigma(a) < \tau; X_0 = y] = \mathcal{O}\left(\frac{1}{(1 - \rho)^2}\right) \quad (3.4)$$

as $\rho \uparrow 1$. Similarly, without conditioning on the value of X_0 ,

$$\sup_{a \geq a_\rho^*} \mathbb{E}[\sigma(a) \mid \sigma(a) < \tau] = \mathcal{O}\left(\frac{1}{(1 - \rho)^2}\right) \quad (3.5)$$

as $\rho \uparrow 1$.

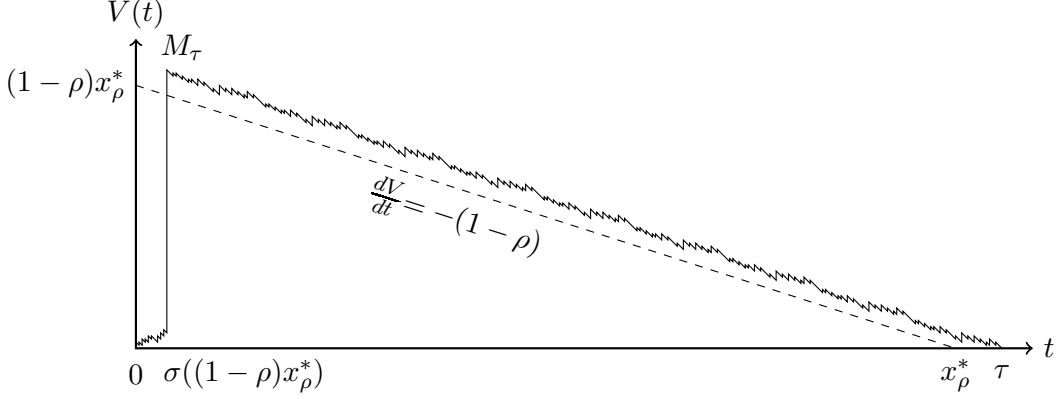


Figure 1: Illustration of an event where $X(t)$ stays positive for a long time. A jump of size $(1 - \rho)x \geq (1 - \rho)x_\rho^*$ happens at time $\sigma((1 - \rho)x) = \mathcal{O}((1 - \rho)^{-2})$. The long-term drift of $-(1 - \rho)$ suggests that $\tau > (1 + o(1))x$.

The above results provide a better understanding of the time at which M_τ is achieved, which will prove essential in the analysis of the stopping time τ itself. The stopping time τ can be related to M_τ in an intuitive way. Assume that M_τ is at least $(1 - \rho)x \geq (1 - \rho)x_\rho^*$ for some large x_ρ^* . Theorem 4 suggests that this level is obtained before or at time $\sigma(M_\tau)$, which is of small order of x . Knowing that the long-term drift of $X(t)$ equals $-(1 - \rho)$, the process will hit level zero approximately at time $\tau = (1 + o(1))x$. An illustration of this process is provided by Figure 1.

This relation between the τ and M_τ was made rigorous by De Meyer and Teugels [29] for regularly varying jump size distributions. Later, Zwart [36] generalized the result to intermediately regularly varying jump size distributions [cf. 10] and general renewal processes $N(t)$, using a sample-path analysis. It is this type of analysis that underlies the proof of our second main result: Theorem 5.

Theorem 5. Suppose $\mathbb{P}(B > x) = L(x)x^{-\alpha}$ for some $\alpha > 2, \alpha \neq 3$ and $L(x)$ slowly varying. For any $k > 1$ define $x_\rho^* := \frac{1}{(1-\rho)^2} \left(\log \frac{1}{1-\rho} \right)^{k\alpha}$. Then

$$\sup_{x \geq x_\rho^*} \left| \mathbb{P}(\tau > x \mid M_\tau > (1 - \rho)x) - 1 \right| \rightarrow 0 \quad (3.6)$$

and

$$\sup_{x \geq x_\rho^*} \left| \mathbb{P}(M_\tau > (1 - \rho)x \mid \tau > x) - 1 \right| \rightarrow 0 \quad (3.7)$$

as $\rho \uparrow 1$. In particular, (3.6) and (3.7) imply

$$\sup_{x \geq x_\rho^*} \left| \frac{\mathbb{P}(\tau > x)}{\mathbb{P}(M_\tau > (1 - \rho)x)} - 1 \right| \rightarrow 0 \quad (3.8)$$

as $\rho \uparrow 1$.

Theorems 3 and 5 together confirm the “single big jump principle”; that is, they suggest that a large time τ is caused by a single large jump. Moreover, Theorem 5 gives an explicit threshold x_ρ^* that discriminates between a small and a large jump, of whom the latter causes the late return to level zero.

Finally, Corollary 1 states that if the compound Poisson process has not hit zero after n jumps, then it has not hit zero after time $n\lambda = n\rho/\mathbb{E}[B]$ jumps and vice versa.

Corollary 1. *Suppose that all conditions in Theorem 5 hold. Then*

$$\sup_{n \geq x_\rho^*} \left| \mathbb{P}(\tilde{N}(\tau) > n \mid \tau > n/\lambda) - 1 \right| \rightarrow 0 \quad (3.9)$$

and

$$\sup_{n \geq x_\rho^*} \left| \mathbb{P}(\tau > n/\lambda \mid \tilde{N}(\tau) > n) - 1 \right| \rightarrow 0 \quad (3.10)$$

as $\rho \uparrow 1$. In particular, (3.9) and (3.10) imply

$$\sup_{n \geq x_\rho^*} \left| \frac{\mathbb{P}(\tilde{N}(\tau) > n)}{\mathbb{P}(\tau > n/\lambda)} - 1 \right| \rightarrow 0 \quad (3.11)$$

as $\rho \uparrow 1$.

The section is concluded by the following remarks. Every next section covers the proof of a theorem.

Remark 1. Relations (3.8) and (3.11) follow directly from their predecessors by observing that

$$\left| \frac{\mathbb{P}(Q)}{\mathbb{P}(R)} - 1 \right| \leq \left| \frac{\mathbb{P}(Q)}{\mathbb{P}(Q; R)} - 1 \right| \frac{\mathbb{P}(Q; R)}{\mathbb{P}(R)} + \left| \frac{\mathbb{P}(Q; R)}{\mathbb{P}(R)} - 1 \right| \quad (3.12)$$

for any two events Q and R with $\mathbb{P}(Q; R) > 0$.

Remark 2. All of the above results easily translate to the M/G/1/FIFO queue with interarrival times A_i , job sizes B_i and traffic intensity ρ . In this case, M_∞ corresponds to the steady state waiting time, M_τ indicates the maximum amount of work in a busy period, B_τ denotes the largest job in a busy period and τ describes the length of a busy period. Of these four random variables, only the waiting time depends on the scheduling discipline. That is, only Theorem 1 is restricted to the FIFO assumption. All other results hold for general scheduling disciplines, regardless of their relation to Theorem 1.

Remark 3. In the M/G/1 setting as described in Remark 2, Theorem 5 states that a busy period of length $x \geq x_\rho^*$ is related to the supremum amount M_τ of work during the busy period. Theorem 3 indicates that this maximum amount is most likely due to a single large job of size $(1 - \rho)x$ that arrives at time $t = \sigma((1 - \rho)x)$. Knowing that the traffic intensity is ρ , one may argue that the server spends a fraction ρ of its time working on jobs that came in after the large job and the remaining $1 - \rho$ fraction of time on the large job (and jobs that were already in the system). The busy period ends when all the work M_τ is served; therefore, we expect the busy period to last until time $\tau \approx \sigma((1 - \rho)x) + x = (1 + o(1))x$.

4 Local asymptotics of the all-time supremum

This section contains the proof of Theorem 1. We consider the all-time supremum by its Pollaczek-Khintchine infinite series representation. From this representation, we distinguish between few jumps and many jumps scenarios (small and large n), where the threshold is approximately $x/\mathbb{E}[B^*]$. It is shown that under the few jumps scenario, a large all-time supremum is most probably due to a large value of a single B_i^* . Contrastingly, the many jumps scenario is shown to be negligible.

Define $S_n^* := \sum_{i=1}^n B_i^*$. The Pollaczek-Khintchine formula for the waiting time distribution is given by [2, Chapter X.9]:

$$\mathbb{P}(M_\infty \in x + \Delta) = \sum_{n=1}^{\infty} (1 - \rho) \rho^n \mathbb{P}(S_n^* \in x + \Delta). \quad (4.1)$$

An equivalent representation of (3.1) is therefore

$$\sup_{x \geq x_\rho} \left| \frac{\sum_{n=1}^{\infty} (1-\rho)\rho^n [\mathbb{P}(S_n^* \in x + \Delta) - n\mathbb{P}(B^* \in x + \Delta)]}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} \right| \rightarrow 0 \quad (4.2)$$

as $\rho \uparrow 1$. Fix a constant δ such that $\max\{\frac{1}{2}, \frac{1}{\alpha-1}\} < \delta < 1$ and define $U_\delta(x) := \lfloor (x - x^\delta)/\mu \rfloor$. Then the numerator in (4.2) can be decomposed as

$$\begin{aligned} & \left| \sum_{n=1}^{\infty} (1-\rho)\rho^n [\mathbb{P}(S_n^* \in x + \Delta) - n\mathbb{P}(B^* \in x + \Delta)] \right| \\ & \leq \sum_{n=1}^{U_\delta(x)} (1-\rho)\rho^n \left| \mathbb{P}(S_n^* \in x + \Delta) - n\mathbb{P}(B^* \in x - (n-1)\mu + \Delta) \right| \\ & \quad + \sum_{n=1}^{U_\delta(x)} (1-\rho)\rho^n n \left| \mathbb{P}(B^* \in x - (n-1)\mu + \Delta) - \mathbb{P}(B^* \in x + \Delta) \right| \\ & \quad + \left| \sum_{n=U_\delta(x)+1}^{\infty} (1-\rho)\rho^n [\mathbb{P}(S_n^* \in x + \Delta) - n\mathbb{P}(B^* \in x + \Delta)] \right|. \end{aligned} \quad (4.3)$$

Here, the first term corresponds to the few jumps scenario and the third term corresponds to the many jumps scenario. The second term corrects a shift in the argument of $\mathbb{P}(B^* \in \cdot)$, which is required for application of Lemma 1. Lemma 1 quantifies the similar behaviour between a single jump with a long remaining service time, and the sum of remaining service times of several jumps.

Lemma 1. *Suppose ξ is a non-negative regularly varying random variable whose distribution function has index $-\alpha_\xi < -2$, $\alpha_\xi \neq 3$; i.e. $\mathbb{P}(\xi > x) = L(x)x^{-\alpha_\xi}$. Let F_{ξ^*} be the integrated tail distribution of ξ with index $-\alpha_\xi + 1 < -1$ and i.i.d. samples $\xi^*, \xi_1^*, \xi_2^*, \dots$. For any $\max\{\frac{1}{\alpha_\xi-1}, \frac{1}{2}\} < \Gamma < 1$ define $U_\Gamma(x) = \lfloor \frac{x-x^\Gamma}{\mathbb{E}[\xi^*]} \rfloor$. Then, there exists a non-increasing function $\phi(x)$ satisfying $\phi(x) \downarrow 0$ as $x \rightarrow \infty$ such that*

$$\sup_{1 \leq n \leq M_\Gamma(x)} \left| \frac{\mathbb{P}(\xi_1^* + \dots + \xi_n^* \in x + \Delta)}{n\mathbb{P}(\xi^* \in x - (n-1)\mathbb{E}[\xi^*] + \Delta)} - 1 \right| \leq \phi(x).$$

Proof. The proof is delayed to the end of this section and relies heavily on the machinery provided by Denisov et al. [14]. \square

Lemma 1 guarantees that, for some non-increasing $\phi(x) \downarrow 0$ as $x \rightarrow \infty$, expression (4.3) is dominated by

$$\begin{aligned} & \phi(x) \sum_{n=1}^{U_\delta(x)} (1-\rho)\rho^n n \mathbb{P}(B^* \in x + \Delta) \\ & + (1 + \phi(x)) \sum_{n=1}^{U_\delta(x)} (1-\rho)\rho^n n \left| \mathbb{P}(B^* \in x - (n-1)\mu + \Delta) - \mathbb{P}(B^* \in x + \Delta) \right| \\ & + \sum_{n=U_\delta(x)+1}^{\infty} (1-\rho)\rho^n [1 + n\mathbb{P}(B^* \in x + \Delta)] \\ & = \phi(x)I + (1 + \phi(x))II + III. \end{aligned}$$

Term I is bounded by $\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)$, so that $x_\rho \rightarrow \infty$ implies

$$\sup_{x \geq x_\rho} \frac{\phi(x)I}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} \leq \phi(x_\rho) \rightarrow 0 \quad (4.4)$$

as $\rho \uparrow 1$.

Error term II is split in two parts. Fix γ such that $0 < \gamma < \delta$ and define $V_\gamma(x) := \lfloor (1 - \gamma)x/\mu \rfloor$. For x sufficiently large we have $V_\gamma(x) < U_\delta(x)$ and II can be written as

$$\begin{aligned} & \sum_{n=1}^{U_\delta(x)} (1 - \rho)\rho^n n \left| \mathbb{P}(B^* \in x - (n - 1)\mu + \Delta) - \mathbb{P}(B^* \in x + \Delta) \right| \\ &= \sum_{n=1}^{V_\gamma(x)} (1 - \rho)\rho^n n \left| \mathbb{P}(B^* \in x - (n - 1)\mu + \Delta) - \mathbb{P}(B^* \in x + \Delta) \right| \\ & \quad + \sum_{n=V_\gamma(x)+1}^{U_\delta(x)} (1 - \rho)\rho^n n \left| \mathbb{P}(B^* \in x - (n - 1)\mu + \Delta) - \mathbb{P}(B^* \in x + \Delta) \right| \\ &= IIa + IIb. \end{aligned}$$

For $1 \leq n \leq V_\gamma(x)$ we have

$$\begin{aligned} \frac{\mathbb{P}(B^* \in x - (n - 1)\mu + \Delta)}{\mathbb{P}(B^* \in x + \Delta)} &\leq \frac{\mathbb{P}(B > x - (n - 1)\mu)}{\mathbb{P}(B > x + T)} \\ &\sim \left(1 - \frac{(n - 1)\mu + T}{x + T} \right)^{-\alpha} \\ &= 1 + \sum_{m=1}^{\infty} \frac{\alpha(\alpha + 1) \cdots (\alpha + m - 1)}{m!} \left(\frac{(n - 1)\mu + T}{x + T} \right)^m \\ &\leq 1 + \alpha \left(\frac{(n - 1)\mu + T}{x + T} \right) \left(1 - \frac{(n - 1)\mu + T}{x + T} \right)^{-\alpha-1} \\ &\lesssim 1 + \alpha\gamma^{-\alpha-1} \frac{(n - 1)\mu + T}{x + T}. \end{aligned}$$

Therefore,

$$\frac{\mathbb{P}(B^* \in x - (n - 1)\mu + \Delta)}{\mathbb{P}(B^* \in x + \Delta)} - 1 \lesssim C \frac{n - 1}{x}$$

as $x \rightarrow \infty$. Substituting this into IIa gives

$$\begin{aligned} IIa &\lesssim C\mathbb{P}(B^* \in x + \Delta) \frac{1}{x} \sum_{n=1}^{V_\gamma(x)} (1 - \rho)\rho^n n(n - 1) \\ &\leq C\mathbb{P}(B^* \in x + \Delta) \frac{2\rho^2}{(1 - \rho)^2 x} \left(1 - \rho^{V_\gamma(x)} - (1 - \rho)V_\gamma(x)\rho^{V_\gamma(x)} \right) \\ &\leq C \frac{\rho}{(1 - \rho)^2 x} \mathbb{P}(B^* \in x + \Delta), \end{aligned}$$

and hence

$$\sup_{x \geq x_\rho} \frac{IIa}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} \lesssim C \frac{1}{\log \frac{1}{1-\rho}} \rightarrow 0 \quad (4.5)$$

as $\rho \uparrow 1$.

Next, we consider term *IIb*. Using the fact that $\mathbb{P}(B^* \in y + \Delta)$ is decreasing in y , term *IIb* is bounded by

$$\begin{aligned} IIb &\leq C(1-\rho)\rho^{V_\gamma(x)+1}x \sum_{n=V_\gamma(x)+1}^{U_\delta(x)} \mathbb{P}(B^* \in x - (n-1)\mu + \Delta) \\ &\leq C(1-\rho)\rho^{(1-\gamma)x/\mu}x \int_{x-\mu U_\delta(x)}^{x-\mu V_\gamma(x)} \mathbb{P}(B^* \in t + \Delta) dt. \end{aligned}$$

Noting that $\mathbb{P}(B^* \in x + \Delta)$ is regularly varying with index $-\alpha < -2$, Theorem 1.5.11 in [6] indicates that

$$\begin{aligned} IIb &\lesssim C(1-\rho)\rho^{(1-\gamma)x/\mu}x(x - \mu U_\delta(x))\mathbb{P}(B^* \in x - \mu U_\delta(x) + \Delta) \\ &\leq C(1-\rho)\rho^{(1-\gamma)x/\mu}x^{1+\delta}\mathbb{P}(B^* \in x^\delta - \Delta). \end{aligned}$$

It remains to verify that *IIb* decreases sufficiently fast for $x \geq x_\rho$. One can see that

$$\begin{aligned} \sup_{x \geq x_\rho} \frac{IIb}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} &\lesssim C \sup_{x \geq x_\rho} (1-\rho)^2 \rho^{(1-\gamma)\frac{x}{\mu}-1} x^{1+\delta} \frac{\mathbb{P}(B^* \in x^\delta + \Delta)}{\mathbb{P}(B^* \in x + \Delta)} \\ &\leq C \sup_{x \geq x_\rho} (1-\rho)^2 \rho^{(1-\gamma)\frac{x}{\mu}-1} x^{1+\delta} \frac{\mathbb{P}(B > x^\delta)}{\mathbb{P}(B > x + T)} \\ &\sim C \sup_{x \geq x_\rho} (1-\rho)^2 e^{((1-\gamma)\frac{x}{\mu}-1) \log \rho} x^{1+\delta+(1-\delta)\alpha} \\ &\leq C \sup_{x \geq x_\rho} (1-\rho)^2 e^{-((1-\gamma)\frac{x}{\mu}-1)(1-\rho)} x^{1+\delta+(1-\delta)\alpha}, \end{aligned} \quad (4.6)$$

where we have used $\log \rho \leq -(1-\rho)$ for all $\rho \geq 0$.

For any differentiable function $g(x)$, the derivative of $e^{g(x)}x^a$ is given by $e^{g(x)}x^{a-1}(xg'(x)+a)$. Hence, the supremum of (4.6) is attained in $x = x_0 := \frac{1}{1-\rho} \frac{\mu}{1-\gamma} (1+\delta+(1-\delta)\alpha)$. For ρ sufficiently close to one we have $x_0 \leq x_\rho$, and therefore

$$\sup_{x \geq x_\rho} \frac{IIb}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} \lesssim C(1-\rho)^2 e^{-(1-\gamma)(1-\rho)\frac{x_\rho}{\mu}} x_\rho^{1+\delta+(1-\delta)\alpha}.$$

Substituting $x_\rho = k\mu(\alpha-1)\frac{1}{1-\rho} \log \frac{1}{1-\rho}$ now gives

$$\begin{aligned} \sup_{x \geq x_\rho} \frac{IIb}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} &\lesssim C(1-\rho)^{k(1-\gamma)(\alpha-1)-(1-\delta)(\alpha-1)} \left(\log \frac{1}{1-\rho} \right)^{1+\delta+(1-\delta)\alpha} \\ &\rightarrow 0 \end{aligned} \quad (4.7)$$

as $\rho \uparrow 1$, since $\gamma < \delta$. This verifies the convergence of term *II*.

We continue with the analysis of term *III*. This term rewritten to two smaller terms:

$$\begin{aligned} III &= \rho^{U_\delta(x)+1} + \left[(U_\delta(x) + 1) \rho^{U_\delta(x)+1} + \frac{\rho^{U_\delta(x)+2}}{1-\rho} \right] \mathbb{P}(B^* \in x + \Delta) \\ &\leq \rho^{\frac{x-x^\delta}{\mu}} + \left[\left(\frac{x-x^\delta}{\mu} + 1 \right) + \frac{\rho}{1-\rho} \right] \mathbb{P}(B^* \in x + \Delta) \rho^{\frac{x-x^\delta}{\mu}} \\ &\leq \rho^{\frac{x-x^\delta}{\mu}} + C \frac{(1-\rho)x+1}{1-\rho} \mathbb{P}(B^* \in x + \Delta) \rho^{\frac{x-x^\delta}{\mu}} \\ &= IIIa + IIIb. \end{aligned}$$

We consider the terms *IIIa* and *IIIb* in order. First assume that $k > 1$. Potter's bound suggests that $\mathbb{P}(B^* \in x + \Delta) \geq T\mathbb{P}(B \geq x + T) \geq TC(x + T)^{-\alpha-\nu}$ for any fixed $\nu > 0$ and x sufficiently large [e.g. 6]. In particular, for $0 < \nu < (k - 1)(\alpha - 1)$,

$$\sup_{x \geq x_\rho} \frac{IIIa}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} \leq \sup_{x \geq x_\rho} C(1 - \rho)(x + T)^{\alpha+\nu} \rho^{\frac{x-x^\delta}{\mu}-1}.$$

The derivative is given by

$$\begin{aligned} \frac{d}{dx}(x + T)^{\alpha+\nu} \rho^{\frac{x-x^\delta}{\mu}-1} &= (x + T)^{\alpha+\nu-1} \rho^{\frac{x-x^\delta}{\mu}-1} \left(\alpha + \nu + \frac{1}{\mu}(x + T)(1 - \delta x^{\delta-1}) \log \rho \right) \\ &\leq (x + T)^{\alpha+\nu-1} \rho^{\frac{x-x^\delta}{\mu}-1} \left(\alpha + \nu - \frac{1}{\mu}(1 - \delta x^{\delta-1})(1 - \rho)x \right) \end{aligned}$$

for x sufficiently large. This implies that the derivative is negative for all $x \geq x_\rho$, provided that ρ is sufficiently close to one. Therefore, the supremum is attained in $x = x_\rho$ and

$$\begin{aligned} \sup_{x \geq x_\rho} \frac{IIIa}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} &\leq C(1 - \rho)e^{(\alpha+\nu) \log x_\rho + \left(\frac{x_\rho - x_\rho^\delta}{\mu} - 1\right) \log \rho + (\alpha+\nu) \log \left\{1 + \frac{T}{x}\right\}} \\ &\leq Ce^{(\alpha+\nu) \log x_\rho - \left(\frac{x_\rho - x_\rho^\delta}{\mu} - 1\right)(1-\rho) - \log \frac{1}{1-\rho} + (\alpha+\nu) \log \left\{1 + \frac{T}{x}\right\}}. \end{aligned}$$

Substitution of $x_\rho = k\mu(\alpha - 1)\frac{1}{1-\rho} \log \frac{1}{1-\rho}$ yields

$$\sup_{x \geq x_\rho} \frac{IIIa}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} \leq C(1 - \rho)e^{(\alpha+\nu-1) \log \frac{1}{1-\rho} - k(\alpha-1) \log \frac{1}{1-\rho} + o\left(\log \frac{1}{1-\rho}\right)}, \quad (4.8)$$

which tends to zero as $\rho \uparrow 1$.

Assuming $k = 1$ and $L(x)/(\log x)^\alpha \rightarrow \infty$ instead, it follows that there exists a non-increasing function $\phi(x) \downarrow 0$ such that $L(x) \geq (\log^\alpha x)/\phi(x)$. Similar to the above analysis we find

$$\begin{aligned} \sup_{x \geq x_\rho} \frac{IIIa}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} &= \sup_{x \geq x_\rho} \frac{1}{L(x + T)} (1 - \rho)(x + T)^\alpha \rho^{\frac{x-x^\delta}{\mu}-1} \\ &\lesssim \phi(x_\rho) \sup_{x \geq x_\rho} \frac{1}{\log^\alpha x} e^{\alpha \log x + \left(\frac{x-x^\delta}{\mu} - 1\right) \log \rho - \log \frac{1}{1-\rho}} \\ &\leq \phi(x_\rho) \frac{1}{\log^\alpha x_\rho} e^{(\alpha-1) \log \frac{1}{1-\rho} + \alpha \log \log \frac{1}{1-\rho} - (\alpha-1)(1-x_\rho^{\delta-1}-x_\rho^{-1}) \log \frac{1}{1-\rho}} \\ &= C\phi(x_\rho) \frac{1}{\log^\alpha \frac{1}{1-\rho}} e^{\alpha \log \log \frac{1}{1-\rho} + (\alpha-1)(x_\rho^{\delta-1}+x_\rho^{-1}) \log \frac{1}{1-\rho}} \\ &= C\phi(x_\rho) e^{(\alpha-1)(x_\rho^{\delta-1}+x_\rho^{-1}) \log \frac{1}{1-\rho}} \\ &\rightarrow 0 \end{aligned}$$

as $\rho \uparrow 1$ since $(\log x)/x^{1-\delta} \rightarrow 0$ for any $\delta < 1$.

Recall that the transition point in Olvera-Cravioto et al. [30] is given by $\tilde{x}_\rho = k\mu(\alpha - 2)\frac{1}{1-\rho} \log \frac{1}{1-\rho}$ and differs by a factor $\frac{\alpha-1}{\alpha-2}$ from x_ρ . This difference is caused by term *IIIa*, which is divided by $\mathbb{P}(B^* > x) \sim x^{1-\alpha}$ in Olvera-Cravioto et al. [30] and by $\mathbb{P}(B^* \in x + \Delta) \sim x^{-\alpha}$ in this article.

Finally, term *IIIb* is bounded as follows:

$$\sup_{x \geq x_\rho} \frac{IIIb}{\frac{\rho}{1-\rho}\mathbb{P}(B^* \in x + \Delta)} = C \sup_{x \geq x_\rho} ((1 - \rho)x + 1) \rho^{\frac{x-x^\delta}{\mu}-1}.$$

Similar to before, it can be shown that the supremum is attained in $x = x_\rho$ for ρ sufficiently close to one. Therefore,

$$\begin{aligned} \sup_{x \geq x_\rho} \frac{IIIb}{\frac{\rho}{1-\rho} \mathbb{P}(B^* \in x + \Delta)} &= C((1-\rho)x_\rho + 1)e^{\left(\frac{x_\rho - x_\rho^\delta}{\mu} - 1\right) \log \rho} \\ &\leq C \log \frac{1}{1-\rho} e^{-k(\alpha-1)(1+o(1)) \log \frac{1}{1-\rho}} \\ &\rightarrow 0 \end{aligned} \tag{4.9}$$

as $\rho \uparrow 1$.

From (4.4), (4.5), (4.7), (4.8) and (4.9), we may conclude that (4.2) and equivalently (3.1) converges to zero. This completes the proof of Theorem 1.

The section is concluded by the proof of Lemma 1.

4.1 Proof of Lemma 1

First consider the case $-\alpha_\xi < -3$. Then $\sigma_{\xi^*}^2 = \mathbb{V}\text{ar}(\xi^*) = \frac{\mathbb{E}[\xi^3]}{3\mathbb{E}[\xi]}$ is finite, and therefore $\bar{\xi}_i^* = \frac{\xi_i^* - \mathbb{E}[\xi^*]}{\sigma_{\xi^*}}$ and $\bar{S}_n^* = \frac{\xi_1^* + \dots + \xi_n^* - n\mathbb{E}[\xi^*]}{\sigma_{\xi^*}}$ are well-defined for all $i \geq 1, n \geq 1$. Since

$$\frac{\mathbb{P}(\xi_1^* + \dots + \xi_n^* \in x + \Delta)}{n\mathbb{P}(\xi^* \in x - (n-1)\mathbb{E}[\xi^*] + \Delta)} = \frac{\mathbb{P}\left(\bar{S}_n^* \in \frac{x - n\mathbb{E}[\xi^*] + \Delta}{\sigma_{\xi^*}}\right)}{n\mathbb{P}\left(\bar{\xi}_1^* \in \frac{x - n\mathbb{E}[\xi^*] + \Delta}{\sigma_{\xi^*}}\right)}, \tag{4.10}$$

the result follows from Theorem 8.1 in Denisov et al. [14] once we show that $(x - n\mathbb{E}[\xi^*])/\sqrt{(\alpha_\xi - 3)n \log n} \rightarrow \infty$ uniformly for $1 \leq n \leq U_\Gamma(x)$. As $\Gamma > \frac{1}{2}$, it follows that

$$\frac{x - n\mathbb{E}[\xi^*]}{\sqrt{(\alpha_\xi - 3)n \log n}} \geq \frac{x - U_\Gamma(x)\mathbb{E}[\xi^*]}{\sqrt{(\alpha_\xi - 3)U_\Gamma(x) \log U_\Gamma(x)}} \sim \sqrt{\frac{\mathbb{E}[\xi^*]}{\alpha_\xi - 3}} x^{\Gamma - \frac{1}{2}}$$

indeed tends to infinity.

Now assume $-3 < -\alpha_\xi < -2$. Let $\tilde{\xi}_i^* = \xi_i^* - \mathbb{E}[\xi^*]$ and $\tilde{S}_n^* = \xi_1^* + \dots + \xi_n^* - n\mathbb{E}[\xi^*]$ for all $i \geq 1, n \geq 1$. Then

$$\frac{\mathbb{P}(\xi_1^* + \dots + \xi_n^* \in x + \Delta)}{n\mathbb{P}(\xi^* \in x - (n-1)\mathbb{E}[\xi^*] + \Delta)} = \frac{\mathbb{P}\left(\tilde{S}_n^* \in x - n\mathbb{E}[\xi^*] + \Delta\right)}{n\mathbb{P}\left(\tilde{\xi}_1^* \in x - n\mathbb{E}[\xi^*] + \Delta\right)}. \tag{4.11}$$

Fix Γ^* such that $\frac{1}{\alpha_\xi - 1} < \Gamma^* < \Gamma$. Theorem 9.1 in Denisov et al. [14] implies that $\mathbb{P}(\tilde{S}_n^* \in x + \Delta) \sim n\mathbb{P}(\tilde{\xi}_1^* \in x + \Delta)$ uniformly for $x \geq n^{\Gamma^*}$. The proof is concluded by showing that $(x - n\mathbb{E}[\xi^*])/n^{\Gamma^*} \rightarrow \infty$ uniformly for $1 \leq n \leq U_\Gamma(x)$, which follows from

$$\frac{x - n\mathbb{E}[\xi^*]}{n^{\Gamma^*}} \geq \frac{x - U_\Gamma(x)\mathbb{E}[\xi^*]}{U_\Gamma(x)^{\Gamma^*}} \sim \mathbb{E}[\xi^*]^{\Gamma^*} x^{\Gamma - \Gamma^*}.$$

Regularly varying distributions with index $-\alpha_\xi = -3$ are not covered explicitly by the work of Denisov et al. [14]. Olvera-Cravioto et al. [30, Lemma 3.2] suggest that all results in this section still hold for the boundary value $\alpha = 3$, but in this case the corresponding analysis “is rather technical and does not provide additional insights.”

5 Asymptotics of the supremum M_τ and jump sizes

This section is dedicated to the proof of Theorem 2. Takács [33, Section 29] and Cohen [11] have independently shown that

$$\mathbb{P}(M_\tau > x) = \frac{1}{\lambda} \frac{d}{dy} \log \mathbb{P}(M_\infty < y) \Big|_{y=x}.$$

Theorem 1 allows us to analyse this right-hand side.

It is known that $\mathbb{P}(M_\infty < x) = (1 - \rho)W_\rho(x)$ for a scale function $W_\rho(x)$, satisfying $W_\rho(x) > 0$ for $x > 0$. The definition and some properties of $W_\rho(x)$ are provided in Section 7. The only property of interest for the current section is that $\frac{d}{dy} \log W_\rho(y)$ is non-increasing and positive. This particularly implies that $\log W_\rho(x)$ is concave and one can apply standard calculus to obtain

$$\begin{aligned} \mathbb{P}(M_\tau > x) &= \frac{1}{\lambda} \frac{d}{dy} \log W_\rho(y) \Big|_{y=x} \\ &\leq \frac{1}{\lambda} [\log W_\rho(x) - \log W_\rho(x-1)] \\ &= \frac{1}{\lambda} \log \frac{W_\rho(x)}{W_\rho(x-1)} \\ &\leq \frac{1}{\lambda} \left[\frac{W_\rho(x)}{W_\rho(x-1)} - 1 \right] \\ &= \frac{1}{\lambda} \frac{\mathbb{P}(M_\infty \in [x-1, x))}{\mathbb{P}(M_\infty < x-1)}. \end{aligned} \tag{5.1}$$

Theorem 1 then implies

$$\mathbb{P}(M_\tau > x) \lesssim \frac{1}{\lambda} \frac{1}{\mathbb{P}(M_\infty < x-1)} \frac{\rho}{1-\rho} \mathbb{P}(B^* \in [x-1, x))$$

for $x \geq x_\rho$. Applying the simple bound $\mathbb{P}(B^* \in [x-1, x)) \leq \frac{1}{\mathbb{E}[B]} \mathbb{P}(B > x-1)$ yields the upper bound

$$\mathbb{P}(M_\tau > x) \lesssim \frac{1}{\mathbb{P}(M_\infty < x-1)} \frac{1}{1-\rho} \mathbb{P}(B > x-1),$$

which, since B is long-tailed, is asymptotically equivalent to

$$\mathbb{P}(M_\tau > x) \lesssim \frac{1}{\mathbb{P}(M_\infty < x-1)} \frac{1}{1-\rho} \mathbb{P}(B > x). \tag{5.2}$$

This concludes the upper bound analysis for $\mathbb{P}(M_\tau > x)$.

Similarly, $\mathbb{P}(M_\tau > x)$ can be bounded from below. Using the inequality $\log x \geq 1 - \frac{1}{x}$ for all $x \geq 0$ and slightly altering the above analysis dictates

$$\mathbb{P}(M_\tau > x) \gtrsim \frac{1}{\mathbb{P}(M_\infty < x+1)} \frac{1}{1-\rho} \mathbb{P}(B > x) \geq \frac{\rho}{1-\rho} \mathbb{P}(B > x). \tag{5.3}$$

for all $x \geq x_\rho$ as $\rho \uparrow 1$.

The asymptotic upper and lower bound quickly conclude the proof:

$$\sup_{x \geq x_\rho} \left| \frac{\mathbb{P}(M_\tau > x)}{\frac{\rho}{1-\rho} \mathbb{P}(B > x)} - 1 \right| \lesssim \sup_{x \geq x_\rho} \frac{1}{\rho \mathbb{P}(M_\infty < x-1)} - 1 \rightarrow 0$$

as $\rho \uparrow 1$.

6 Asymptotics of the supremum M_τ and supremum jump size

This section contributes the proof of Theorem 3. The following equality is an interpretation of expression (3.4) in Boxma [9]:

$$\mathbb{P}(B_\tau > x) = \mathbb{P}(B > x) + \int_0^x \left[1 - e^{-\lambda \mathbb{P}(B_\tau > x)t} \right] d\mathbb{P}(B > t), \quad (6.1)$$

From this equality it follows that

$$\begin{aligned} \frac{\mathbb{P}(B > x)}{\mathbb{P}(B_\tau > x)} &= 1 - \int_0^x \left[\frac{1 - e^{-\lambda \mathbb{P}(B_\tau > x)t}}{\mathbb{P}(B_\tau > x)} \right] d\mathbb{P}(B > t) \\ &= 1 - \lambda \int_0^x t d\mathbb{P}(B > t) \\ &\quad + \lambda \int_0^x \left[1 - \frac{1 - e^{-\lambda \mathbb{P}(B_\tau > x)t}}{\lambda \mathbb{P}(B_\tau > x)t} \right] t d\mathbb{P}(B > t) \\ &= 1 - \rho + \lambda \int_x^\infty t d\mathbb{P}(B > t) \\ &\quad + \lambda \int_0^x \left[1 - \frac{1 - e^{-\lambda \mathbb{P}(B_\tau > x)t}}{\lambda \mathbb{P}(B_\tau > x)t} \right] t d\mathbb{P}(B > t), \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{1-\rho} \frac{\mathbb{P}(B > x)}{\mathbb{P}(B_\tau > x)} - 1 &= \frac{\lambda}{1-\rho} \int_x^\infty t d\mathbb{P}(B > t) \\ &\quad + \frac{\lambda}{1-\rho} \int_0^x \left[1 - \frac{1 - e^{-\lambda \mathbb{P}(B_\tau > x)t}}{\lambda \mathbb{P}(B_\tau > x)t} \right] t d\mathbb{P}(B > t). \end{aligned} \quad (6.2)$$

Note that the right-hand side of the latter expression is non-negative because $(1 - e^{-y})/y \leq 1$.

Consider the first integral in (6.2). It can be bounded as follows:

$$\begin{aligned} \frac{\lambda}{1-\rho} \int_x^\infty t d\mathbb{P}(B > t) &= \frac{\lambda}{1-\rho} \mathbb{E}[B \mathbb{1}\{B > x\}] \\ &= \frac{\lambda}{1-\rho} \mathbb{E}\left[B - x \mid B > x\right] \mathbb{P}(B > x) + \frac{\lambda}{1-\rho} x \mathbb{P}(B > x) \\ &\leq C \frac{\lambda}{1-\rho} x \mathbb{P}(B > x), \end{aligned}$$

since $\mathbb{E}[B - x \mid B > x] \sim \frac{x}{\alpha-1}$, as shown in Embrechts et al. [16, p.162]. From some $0 < \nu \leq \alpha - 2$, Potter's bound yields

$$\begin{aligned} \sup_{x \geq x_\rho} \frac{\lambda}{1-\rho} \int_x^\infty t d\mathbb{P}(B > t) &\leq \sup_{x \geq x_\rho} C \frac{1}{1-\rho} x^{-\alpha+\nu+1} \\ &\leq C \frac{1}{(1-\rho)x} \\ &= C \frac{1}{\log \frac{1}{1-\rho}} \\ &\rightarrow 0 \end{aligned} \quad (6.3)$$

as $\rho \uparrow 1$. This bounds the first integral.

Consider the second integral in (6.2). The bound $e^y \geq 1 + y + y^2/2$ for $y \geq 0$ implies

$$\begin{aligned}
& \sup_{x \geq x_\rho} \frac{\lambda}{1-\rho} \int_0^x \left[1 - \frac{1 - e^{-\lambda \mathbb{P}(B_\tau > x)t}}{\lambda \mathbb{P}(B_\tau > x)t} \right] t \, d\mathbb{P}(B > t) \\
& \leq \sup_{x \geq x_\rho} \frac{\lambda}{1-\rho} \int_0^x \left[1 - \frac{1 - \frac{1}{1 + \lambda \mathbb{P}(B_\tau > x)t + \lambda^2 \mathbb{P}(B_\tau > x)^2 t^2 / 2}}{\lambda \mathbb{P}(B_\tau > x)t} \right] t \, d\mathbb{P}(B > t) \\
& = \sup_{x \geq x_\rho} \frac{\lambda}{2(1-\rho)} \int_0^x \frac{\lambda \mathbb{P}(B_\tau > x)t + \lambda^2 \mathbb{P}(B_\tau > x)^2 t^2}{1 + \lambda \mathbb{P}(B_\tau > x)t + \lambda^2 \mathbb{P}(B_\tau > x)^2 t^2 / 2} t \, d\mathbb{P}(B > t) \\
& \leq \sup_{x \geq x_\rho} \frac{\lambda^2}{2} \frac{\mathbb{P}(B_\tau > x)}{1-\rho} \int_0^x [t^2 + \lambda \mathbb{P}(B_\tau > x)t^3] \, d\mathbb{P}(B > t) \\
& \leq \sup_{x \geq x_\rho} \frac{\lambda^2}{2} \frac{\mathbb{P}(B_\tau > x)}{1-\rho} [1 + \lambda \mathbb{P}(B_\tau > x)x] \mathbb{E}[B^2] \\
& \leq \sup_{x \geq x_\rho} \frac{\lambda^2}{2} \frac{\mathbb{P}(M_\tau > x)}{1-\rho} [1 + \lambda \mathbb{P}(M_\tau > x)x] \mathbb{E}[B^2].
\end{aligned}$$

Theorem 2 states that $\mathbb{P}(M_\tau > x) \sim \frac{\rho}{1-\rho} \mathbb{P}(B > x)$ uniformly for $x \geq x_\rho$. Applying this to the above expression and using Potter's bound yields

$$\begin{aligned}
\sup_{x \geq x_\rho} \frac{\lambda}{1-\rho} \int_0^x \left[1 - \frac{1 - e^{-\lambda \mathbb{P}(B_\tau > x)t}}{\lambda \mathbb{P}(B_\tau > x)t} \right] t \, d\mathbb{P}(B > t) & \lesssim \sup_{x \geq x_\rho} C \frac{\mathbb{P}(B > x)}{(1-\rho)^2} \left[1 + \frac{1}{1-\rho} \mathbb{P}(B > x)x \right] \\
& \leq \sup_{x \geq x_\rho} C \frac{x^{-\alpha+\nu}}{(1-\rho)^2} \left[1 + \frac{x^{1-\alpha+\nu}}{1-\rho} \right] \\
& = C \frac{1}{\log^2 \frac{1}{1-\rho}} \left[1 + \frac{1}{\log \frac{1}{1-\rho}} \right]. \tag{6.4}
\end{aligned}$$

This bounds the second integral.

From (6.2), (6.3), (6.4) and Theorem 2, it follows that

$$\begin{aligned}
\sup_{x \geq x_\rho} \left| \frac{\mathbb{P}(M_\tau > x)}{\mathbb{P}(B_\tau > x)} - 1 \right| & \leq \sup_{x \geq x_\rho} \left| \frac{\mathbb{P}(M_\tau > x)}{\frac{\rho}{1-\rho} \mathbb{P}(B > x)} - 1 \right| \\
& \quad + \sup_{x \geq x_\rho} \left| \frac{\mathbb{P}(M_\tau > x)}{\frac{\rho}{1-\rho} \mathbb{P}(B > x)} \right| \cdot \sup_{x \geq x_\rho} \left| \frac{\frac{\rho}{1-\rho} \mathbb{P}(B > x)}{\mathbb{P}(B_\tau > x)} - 1 \right| \\
& \rightarrow 0 \tag{6.5}
\end{aligned}$$

as $\rho \uparrow 1$.

7 Asymptotics of the conditional expectation of the passage time of level a

This section is dedicated to the proof of Theorem 4. Theorem 4 regards the expected first passage time of level a , $\sigma(a)$, provided that level a is reached before level 0: $\sigma(a) < \tau$. In particular, we consider high levels $a \geq a_\rho^* := 2k\mu(\alpha - 1) \frac{1}{1-\rho} \log \frac{1}{1-\rho}$ for any $k > 1$. The theorem considers two different scenarios. In the first scenario, we condition on the initial value $X(0) = y$. In the second scenario, the initial value $X(0)$ is a random variable with the same distribution as a general jump size B . The analysis for this latter scenario is based on the following decomposition:

$$\mathbb{E}[\sigma(a) \mid \sigma(a) < \tau] = \int_0^a \mathbb{E}[\sigma(a) \mid \sigma(a) < \tau] \, d\mathbb{P}(B \leq y). \tag{7.1}$$

That is, we condition the former expectation to the initial value of the process and integrate over all possible initial values. A distinction is made between a “small” and a “large” random initial value; a precise definition of which is given at the end of these introductory paragraphs. The first scenario, where the initial value is fixed, is implicit in the analysis of a small random initial value, and its proof is concluded at the end of Section 7.1.

The derivation of results in this section relies heavily on the theory of spectrally one-sided Lévy processes and q -scale functions, e.g. as documented by Kyprianou [26]. Our interest in q -scale functions $W_\rho^{(q)}$ originates from the close connection between the all-time supremum M_∞ and the 0-scale function $W_\rho(x) := W_\rho^{(0)}(x)$. Of particular importance is the relation

$$\mathbb{P}(M_\infty < x) = \frac{1}{1 - \rho} W_\rho(x), \quad (7.2)$$

which can be derived from Corollary IX.3.4 in Asmussen [2] [e.g. as shown in 5]. Prior to stating the definition of $W^{(q)}$, we define the Laplace exponent $\psi(\lambda) := \frac{1}{t} \log \mathbb{E}(e^{-\lambda X(t)})$ and the right inverse $\phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$. Now, for every $q \geq 0$ the q -scale function $W_\rho^{(q)}(x) : \mathbb{R} \rightarrow [0, \infty)$ corresponding to the spectrally positive Lévy process $X(t)$ is defined on $x < 0$ as $W^{(q)}(x) = 0$, and on $x \geq 0$ by its Laplace transform:

$$\int_0^\infty e^{-\beta x} W_\rho^{(q)}(x) dx = \frac{1}{\psi(\beta) - q} \text{ for } \beta > \phi(q). \quad (7.3)$$

Additionally, Kyprianou gives a representation of $W_\rho^{(q)}(x)$ in terms of $W_\rho(x)$ in his equation (8.29):

$$W_\rho^{(q)}(x) = \sum_{k \geq 0} q^k W_\rho^{(k+1) \circledast}(x), \quad (7.4)$$

where the operator $f \circledast g(x)$ denotes the convolution $\int_0^x f(x-y)g(y) dy$ and $f^{k \circledast}(x)$ is the k -fold convolution of f with itself.

An alternative representation of $W_\rho(x)$ is provided by Kuznetsov et al., stating that for all $x < b$, some measure n_ρ and a real-valued function $\bar{\xi}_\rho$,

$$W_\rho(x) = W_\rho(b) \exp \left(- \int_x^b n_\rho(\bar{\xi}_\rho > t) dt \right). \quad (7.5)$$

This representation directly implies that

$$\frac{f_M(x)}{\mathbb{P}(M_\infty < x)} := \frac{\frac{d}{dy} \mathbb{P}(M_\infty < y) \Big|_{y=x}}{\mathbb{P}(M_\infty < x)} = \frac{d}{dy} \log W_\rho(y) \Big|_{y=x} = n_\rho(\bar{\xi}_\rho > x) \quad (7.6)$$

is non-increasing.

For the remainder of this section, the subscripts ρ for $W_\rho(x)$ and $W_\rho^{(q)}(x)$ are discarded. We also introduce the short-hand notations $\mathbb{E}_x[\cdot]$ and $\mathbb{P}_x(\cdot)$ for the conditional expectation $\mathbb{E}[\cdot | X(0) = x]$ and conditional probability $\mathbb{P}(\cdot | X(0) = x)$, respectively.

Let $Z_\rho^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) dy$. From (7.4) and the spectrally *positive* Lévy Process

interpretation of Theorem 8.1 in Kyprianou [26], it follows that

$$\begin{aligned}
\mathbb{E}_y[\sigma(a) \mathbb{1}\{\sigma(a) < \tau\}] &= -\frac{d}{dq} \mathbb{E}_y[e^{-q\sigma(a)} \mathbb{1}\{\sigma(a) < \tau\}] \Big|_{q=0} \\
&= -\frac{d}{dq} Z_\rho^{(q)}(a-y) + \frac{W^{(q)}(a-y)}{W^{(q)}(a)} \frac{d}{dq} Z_\rho^{(q)}(a) \\
&\quad + Z_\rho^{(q)}(a) \frac{W^{(q)}(a) \frac{d}{dq} W^{(q)}(a-y) - W^{(q)}(a-y) \frac{d}{dq} W^{(q)}(a)}{(W^{(q)}(a))^2} \Big|_{q=0} \\
&= -\int_0^{a-y} W(t) dt + \frac{W(a-y)}{W(a)} \int_0^a W(t) dt \\
&\quad + \frac{W(a)W^{2*}(a-y) - W(a-y)W^{2*}(a)}{(W(a))^2} \\
&= \frac{1}{W(a)} \frac{W(a)W^{2*}(a-y) - W(a-y)W^{2*}(a)}{W(a)} \\
&\quad + \frac{W(a-y) \int_0^a W(t) dt - W(a) \int_0^{a-y} W(t) dt}{W(a)}.
\end{aligned}$$

Using the identity $\mathbb{P}_y(\sigma(a) < \tau) = \frac{W(a)-W(a-y)}{W(a)}$ gives a representation of the conditional expectation in terms of scale functions:

$$\begin{aligned}
\mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau] &= \frac{1}{W(a)} \frac{W(a)W^{2*}(a-y) - W(a-y)W^{2*}(a)}{W(a) - W(a-y)} \\
&\quad + \frac{W(a-y) \int_0^a W(t) dt - W(a) \int_0^{a-y} W(t) dt}{W(a) - W(a-y)} \\
&= \frac{1}{W(a)} \frac{W(a) \int_0^{a-y} W(a-y-t)W(t) dt - W(a-y) \int_0^a W(a-t)W(t) dt}{W(a) - W(a-y)} \\
&\quad + \frac{1}{W(a)} \frac{W(a-y) \int_0^a W(a)W(t) dt - W(a) \int_0^{a-y} W(a)W(t) dt}{W(a) - W(a-y)} \\
&= \frac{W(a-y)}{W(a)} \frac{\int_0^a (W(a) - W(a-t))W(t) dt}{W(a) - W(a-y)} \\
&\quad - \frac{\int_0^{a-y} (W(a) - W(a-y-t))W(t) dt}{W(a) - W(a-y)}.
\end{aligned}$$

Substitute (7.2) into the above expression to obtain

$$\begin{aligned}
\mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau] &= \frac{\mathbb{P}(M_\infty < a-y)}{\mathbb{P}(M_\infty < a)} \frac{\int_0^a \mathbb{P}(M_\infty \in [a-t, a]) \mathbb{P}(M_\infty < t) dt}{(1-\rho) \mathbb{P}(M_\infty \in [a-y, a])} \\
&\quad - \frac{\int_0^{a-y} \mathbb{P}(M_\infty \in [a-y-t, a]) \mathbb{P}(M_\infty < t) dt}{(1-\rho) \mathbb{P}(M_\infty \in [a-y, a])}. \quad (7.7)
\end{aligned}$$

The analysis of this expression depends on the initial value. We distinguish two categories of initial values: small and large values. Fix d such that $0 < d < 1 - \frac{1}{k} < 1$. Small values are of size at most $d \cdot a$, all other values are large values.

7.1 Small random initial value or fixed initial value

This section considers the process from a small initial value y , i.e. $y \leq da$. Discarding the first ratio in (7.7) gives the following upper bound:

$$\begin{aligned} \mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau] &\leq \frac{\int_{a-y}^a \mathbb{P}(M_\infty \in [a-t, a)) \mathbb{P}(M_\infty < t) dt - \int_0^{a-y} \mathbb{P}(M_\infty \in [a-y-t, a-t)) \mathbb{P}(M_\infty < t) dt}{(1-\rho) \mathbb{P}(M_\infty \in [a-y, a))} \\ &= \frac{K_{num}(y, a)}{K_{denom}(y, a)}. \end{aligned}$$

For any y -differentiable function $G(y, a)$, it is known that $G(y, a) = G(0, a) + \int_0^y \frac{d}{ds} G(s, a) \big|_{s=t} dt$. Let $M_\infty^{(i)}, i = 1, 2$ be independent copies of M_∞ . Taking the derivative of K_{num} with respect to y yields

$$\begin{aligned} \frac{d}{dy} K_{num}(y, a) &= \mathbb{P}(M_\infty^{(2)} \in [y, a)) \mathbb{P}(M_\infty^{(1)} < a-y) + \mathbb{P}(M_\infty^{(2)} < y) \mathbb{P}(M_\infty^{(1)} < a-y) \\ &\quad - \int_0^{a-y} \mathbb{P}(M_\infty^{(1)} < t) d\mathbb{P}(M_\infty^{(2)} < a-y-t) \\ &= \mathbb{P}(M_\infty^{(2)} < a) \mathbb{P}(M_\infty^{(1)} < a-y) - \mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} < a-y) \\ &= \int_0^y \mathbb{P}(M_\infty^{(2)} < a) \mathbb{P}(M_\infty^{(1)} < a-z) dz \\ &\quad - \int_0^y \mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} < a-z) dz \\ &= \int_0^y \mathbb{P}(M_\infty^{(1)} < a-z) - \mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} < a-z) dz \\ &\quad - \int_0^y \mathbb{P}(M_\infty^{(2)} \geq a) \mathbb{P}(M_\infty^{(1)} < a-z) dz \\ &= \int_0^y \mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} \geq a-z; M_\infty^{(1)} < a-z) dz \\ &\quad - \int_0^y \mathbb{P}(M_\infty^{(2)} \geq a-z) \mathbb{P}(M_\infty^{(1)} < a-z) dz \\ &\quad + \int_0^y \mathbb{P}(M_\infty^{(2)} \in [a-z, a)) \mathbb{P}(M_\infty^{(1)} < a-z) dz \\ &= \int_0^y \mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} \geq a-z; M_\infty^{(1)} < a-z; M_\infty^{(2)} < a-z) dz \\ &\quad + \int_0^y \mathbb{P}(M_\infty^{(2)} \in [a-z, a)) \mathbb{P}(M_\infty^{(1)} < a-z) dz, \end{aligned}$$

so that, noting that $K_{num}(0, a) = 0$,

$$\begin{aligned} \frac{K_{num}(y, a)}{K_{denom}(y, a)} &= \frac{\int_0^y \mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} \geq a-z; M_\infty^{(1)} < a-z; M_\infty^{(2)} < a-z) dz}{(1-\rho) \mathbb{P}(M_\infty \in [a-y, a))} \\ &\quad + \frac{\int_0^y \mathbb{P}(M_\infty^{(2)} \in [a-z, a)) \mathbb{P}(M_\infty^{(1)} < a-z) dz}{(1-\rho) \mathbb{P}(M_\infty \in [a-y, a))} \\ &\leq \frac{\int_0^y \mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} \geq a-z; M_\infty^{(1)} < a-z; M_\infty^{(2)} < a-z) dz}{(1-\rho) \mathbb{P}(M_\infty \in [a-y, a))} + \frac{y}{1-\rho} \\ &=: K(y, a) + \frac{y}{1-\rho}. \end{aligned}$$

Consider the integrand of term $K(y, a)$. Let $u = a - z$. By symmetry, we have

$$\begin{aligned}\mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} \geq u; M_\infty^{(1)} < u; M_\infty^{(2)} < u) &\leq 2\mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} \geq u; u/2 \leq M_\infty^{(1)} < u) \\ &= 2 \int_{u/2}^u \mathbb{P}(M_\infty < v) \frac{f_M(v)}{\mathbb{P}(M_\infty < v)} \mathbb{P}(M_\infty \geq u - v) dv,\end{aligned}$$

so that (7.6) implies

$$\begin{aligned}\mathbb{P}(M_\infty^{(1)} + M_\infty^{(2)} \geq u; M_\infty^{(1)} < u; M_\infty^{(2)} < u) &\leq 2\mathbb{P}(M_\infty < u) \frac{f_M(u/2)}{\mathbb{P}(M_\infty < u/2)} \mathbb{E}[M_\infty] \\ &\sim 2f_M(u/2) \mathbb{E}[M_\infty]\end{aligned}$$

as $u \rightarrow \infty$. Substituting this into term $K(y, a)$ and using the fact that $\mathbb{E}[M_\infty] = \frac{\rho}{1-\rho} \frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}$ yields

$$\begin{aligned}K(y, a) &\lesssim 2\mathbb{E}[M_\infty] \frac{\int_0^y f_M((a-z)/2) dz}{(1-\rho)\mathbb{P}(M_\infty \in [a-y, a])} \\ &\leq \frac{C}{(1-\rho)^2} \frac{\mathbb{P}(M_\infty \in [\frac{a-y}{2}, \frac{a}{2}))}{\mathbb{P}(M_\infty \in [a-y, a])}.\end{aligned}$$

Both local probabilities can be represented as a sum of local probabilities over an interval with fixed length. Subsequently, Theorem 1 is applied to bound the above ratio. Fix $y_{\min} > 0$ and consider $y_{\min} \leq y \leq da$. For $S := y_{\min}/2$, we have

$$\frac{\mathbb{P}(M_\infty \in [\frac{a-y}{2}, \frac{a}{2}))}{\mathbb{P}(M_\infty \in [a-y, a])} \leq \frac{\sum_{i=0}^{\lceil \frac{y}{2S}-1 \rceil} \mathbb{P}(M_\infty \in [\frac{a-y}{2} + iS, \frac{a-y}{2} + (i+1)S))}{\sum_{i=0}^{\lfloor \frac{y}{S}-1 \rfloor} \mathbb{P}(M_\infty \in [a-y + iS, a-y + (i+1)S))}.$$

Theorem 1 and our choice for d ensure that, for some non-increasing function $\phi(x) \downarrow 0$,

$$\begin{aligned}\frac{\mathbb{P}(M_\infty \in [\frac{a-y}{2}, \frac{a}{2}))}{\mathbb{P}(M_\infty \in [a-y, a])} &\leq \frac{1 + \phi(\frac{a}{2k})}{1 - \phi(\frac{a}{k})} \frac{\sum_{i=0}^{\lceil \frac{y}{2S}-1 \rceil} \mathbb{P}(B^* \in [\frac{a-y}{2} + iS, \frac{a-y}{2} + (i+1)S))}{\sum_{i=0}^{\lfloor \frac{y}{S}-1 \rfloor} \mathbb{P}(B^* \in [a-y + iS, a-y + (i+1)S))} \\ &\leq \frac{1 + \phi(\frac{a}{2k})}{1 - \phi(\frac{a}{k})} \frac{\frac{y}{2S} + 1}{\frac{y}{S} - 1} \frac{\mathbb{P}(B > \frac{a-y}{2})}{\mathbb{P}(B > a)} \\ &\sim \frac{1 + \frac{2S}{y}}{2 - \frac{2S}{y}} \left(\frac{a-y}{2a} \right)^{-\alpha} \\ &\leq 2(2k)^\alpha.\end{aligned}$$

Now consider $0 < y < y_{\min}$. Relation (7.6) implies

$$\begin{aligned}\frac{\mathbb{P}(M_\infty \in [\frac{a-y}{2}, \frac{a}{2}))}{\mathbb{P}(M_\infty \in [a-y, a])} &\leq \frac{y \sup_{z \in (0, y)} \frac{f_M(\frac{a-z}{2})}{\mathbb{P}(M_\infty < \frac{a-z}{2})} \mathbb{P}(M_\infty < \frac{a-z}{2})}{2y \inf_{z \in (0, y)} \frac{f_M(a-z)}{\mathbb{P}(M_\infty < a-z)} \mathbb{P}(M_\infty < a-z)} \\ &\leq \frac{\frac{f_M(\frac{a-y}{2})}{\mathbb{P}(M_\infty < \frac{a-y}{2})} \mathbb{P}(M_\infty < \frac{a}{2})}{2 \frac{f_M(a)}{\mathbb{P}(M_\infty < a)} \mathbb{P}(M_\infty < a-y)} \\ &= \frac{f_M(\frac{a-y}{2})}{2f_M(a)} \frac{\mathbb{P}(M_\infty < \frac{a}{2}) \mathbb{P}(M_\infty < a)}{\mathbb{P}(M_\infty < \frac{a-y}{2}) \mathbb{P}(M_\infty < a-y)} \\ &\sim \frac{f_M(\frac{a-y}{2})}{2f_M(a)}\end{aligned}$$

as $a \rightarrow \infty$. We conclude that

$$K(y, a) \lesssim \frac{C}{(1-\rho)^2} \left(1 + \mathbb{1}\{y \leq y_{\min}\} \frac{f_M\left(\frac{a-y}{2}\right)}{f_M(a)} \right). \quad (7.8)$$

By taking y_{\min} sufficiently small, the first part of Theorem 4 follows from

$$\begin{aligned} \sup_{a \geq a_\rho^*} \mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau] &\leq \sup_{a \geq a_\rho^*} K(y, a) + \frac{y}{1-\rho} \\ &\lesssim \mathcal{O}\left(\frac{1}{(1-\rho)^2} (1 + (1-\rho)y)\right). \end{aligned}$$

We now continue the analysis for a large random initial value.

7.2 Large random initial value

Complementary to the previous section, we now consider a large initial value, i.e. $da \leq y < a$. Recall that equation (7.7) stated

$$\begin{aligned} \mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau] &= \frac{\mathbb{P}(M_\infty < a-y) \int_0^a \mathbb{P}(M_\infty \in [a-t, a]) \mathbb{P}(M_\infty < t) dt}{\mathbb{P}(M_\infty < a) (1-\rho) \mathbb{P}(M_\infty \in [a-y, a])} \\ &\quad - \frac{\mathbb{P}(M_\infty < a) \int_0^{a-y} \mathbb{P}(M_\infty \in [a-y-t, a]) \mathbb{P}(M_\infty < t) dt}{\mathbb{P}(M_\infty < a) (1-\rho) \mathbb{P}(M_\infty \in [a-y, a])}. \quad (7.7, \text{ revisited}) \end{aligned}$$

Let M_∞^* be the integrated tail distribution of M_∞ , that is, $\frac{d}{dx} \mathbb{P}(M_\infty^* < x) = \mathbb{P}(M_\infty \geq x) / \mathbb{E}[M_\infty]$. Using $\mathbb{P}(M_\infty < t) = 1 - \mathbb{P}(M_\infty \geq t)$ and $\int_0^a \mathbb{P}(M_\infty \in [a-t, a]) dt = \mathbb{E}[M_\infty \mathbb{1}\{M_\infty < a\}]$, we find

$$\begin{aligned} \mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau] &= \frac{\mathbb{P}(M_\infty < a-y) \mathbb{E}[M_\infty \mathbb{1}\{M_\infty < a\}]}{(1-\rho) \mathbb{P}(M_\infty < a) \mathbb{P}(M_\infty \in [a-y, a])} \\ &\quad - \frac{\mathbb{P}(M_\infty < a-y) \mathbb{E}[M_\infty] \int_0^a \mathbb{P}(M_\infty \in [a-t, a]) d\mathbb{P}(M_\infty^* < t)}{(1-\rho) \mathbb{P}(M_\infty < a) \mathbb{P}(M_\infty \in [a-y, a])} \\ &\quad - \frac{\mathbb{P}(M_\infty < a) \mathbb{E}[M_\infty \mathbb{1}\{M_\infty < a-y\}]}{(1-\rho) \mathbb{P}(M_\infty < a) \mathbb{P}(M_\infty \in [a-y, a])} \\ &\quad + \frac{\mathbb{P}(M_\infty < a) \mathbb{E}[M_\infty] \int_0^{a-y} \mathbb{P}(M_\infty \in [a-y-t, a]) d\mathbb{P}(M_\infty^* < t)}{(1-\rho) \mathbb{P}(M_\infty < a) \mathbb{P}(M_\infty \in [a-y, a])} \\ &= \frac{\mathbb{P}(M_\infty < a-y) \mathbb{E}[M_\infty \mathbb{1}\{M_\infty \in [a-y, a]\}]}{(1-\rho) \mathbb{P}(M_\infty < a) \mathbb{P}(M_\infty \in [a-y, a])} \\ &\quad - \frac{\mathbb{P}(M_\infty \in [a-y, a]) \mathbb{E}[M_\infty \mathbb{1}\{M_\infty < a-y\}]}{(1-\rho) \mathbb{P}(M_\infty < a) \mathbb{P}(M_\infty \in [a-y, a])} \\ &\quad - \frac{\mathbb{P}(M_\infty < a-y) \mathbb{E}[M_\infty] \mathbb{P}(M_\infty \in [a-M_\infty^*, a]; M_\infty^* < a)}{(1-\rho) \mathbb{P}(M_\infty < a) \mathbb{P}(M_\infty \in [a-y, a])} \\ &\quad + \frac{\mathbb{P}(M_\infty < a) \mathbb{E}[M_\infty] \mathbb{P}(M_\infty \in [a-y-M_\infty^*, a]; M_\infty^* < a-y)}{(1-\rho) \mathbb{P}(M_\infty < a) \mathbb{P}(M_\infty \in [a-y, a])} \\ &\leq \frac{\mathbb{P}(M_\infty < a-y)}{\mathbb{P}(M_\infty < a)} \frac{\mathbb{E}[M_\infty \mid M_\infty \in [a-y, a]]}{1-\rho} + \frac{\mathbb{E}[M_\infty]}{(1-\rho) \mathbb{P}(M_\infty \in [a-y, a])} \\ &\leq \frac{a}{1-\rho} + \frac{\mathbb{E}[M_\infty]}{(1-\rho) \mathbb{P}(M_\infty \in [a-y, a])}. \end{aligned}$$

Using the bound $y \geq da$ and applying Theorem 1 gives

$$\begin{aligned}
\mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau] &\leq \frac{a}{1-\rho} + \frac{\mathbb{E}[M_\infty]}{(1-\rho)\mathbb{P}(M_\infty \in [(1-d)a, a))} \\
&\leq \frac{a}{1-\rho} + \frac{\mathbb{E}[M_\infty]}{(1-\rho)\sum_{i=0}^{\lfloor da-1 \rfloor} \mathbb{P}(M_\infty \in [(1-d)a+i, (1-d)a+i+1))} \\
&\sim \frac{a}{1-\rho} + \frac{C}{(1-\rho)\sum_{i=0}^{\lfloor da-1 \rfloor} \mathbb{P}(B_\infty^* \in [(1-d)a+i, (1-d)a+i+1))} \\
&\lesssim \frac{a}{1-\rho} + \frac{C}{(1-\rho)a\mathbb{P}(B > a)}. \tag{7.9}
\end{aligned}$$

This completes the analysis of the conditional expectation for large initial values.

7.3 Synthesis of small and large random initial value

From equation (7.1), (7.8) and (7.9) one can deduce that

$$\begin{aligned}
\sup_{a \geq a_\rho^*} \mathbb{E}[\sigma(a) \mid \sigma(a) < \tau] &\leq \sup_{a \geq a_\rho^*} \int_0^{da} \mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau_0] d\mathbb{P}(B < y) \\
&\quad + \sup_{a \geq a_\rho^*} \int_{da}^a \mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau_0] d\mathbb{P}(B < y) \\
&\leq \sup_{a \geq a_\rho^*} \int_0^{da} K(y, a) d\mathbb{P}(B < y) + \sup_{a \geq a_\rho^*} \frac{1}{1-\rho} \int_0^{da} y d\mathbb{P}(B < y) \\
&\quad + \sup_{a \geq a_\rho^*} \mathbb{P}(B \geq da) \sup_{y \in [da, a)} \mathbb{E}_y[\sigma(a) \mid \sigma(a) < \tau] \\
&\lesssim \sup_{a \geq a_\rho^*} \frac{C}{(1-\rho)^2} \left(1 + \int_0^{y_{\min}} \frac{f_M\left(\frac{a-y}{2}\right)}{f_M(a)} d\mathbb{P}(B < t) \right) \\
&\quad + \frac{\mathbb{E}[B]}{1-\rho} + \sup_{a \geq a_\rho^*} \frac{a}{1-\rho} \mathbb{P}(B \geq da) + \sup_{a \geq a_\rho^*} \frac{C}{(1-\rho)a} \frac{\mathbb{P}(B \geq da)}{\mathbb{P}(B \geq a)}. \tag{7.10}
\end{aligned}$$

The fact that $\frac{f_M(x)}{\mathbb{P}(M_\infty \leq x)}$ is decreasing in x implies that the integral in the first term is ultimately bounded by constant:

$$\begin{aligned}
\int_0^{y_{\min}} \frac{f_M\left(\frac{a-y}{2}\right)}{f_M(a)} d\mathbb{P}(B < y) &\leq \frac{\mathbb{P}(M_\infty \leq \frac{a}{2})}{f_M(a)} \int_0^{y_{\min}} \frac{f_M\left(\frac{a-y}{2}\right)}{\mathbb{P}(M_\infty \leq \frac{a-y}{2})} d\mathbb{P}(B < y) \\
&\leq \mathbb{P}(B \in (0, y_{\min})) \frac{\mathbb{P}(M_\infty \leq \frac{a}{2})}{\mathbb{P}(M_\infty \leq a)} \frac{\mathbb{P}(M_\infty \leq a)}{f_M(a)} \frac{f_M\left(\frac{a-y_{\min}}{2}\right)}{\mathbb{P}(M_\infty \leq \frac{a-y_{\min}}{2})} \\
&= C \frac{\mathbb{P}(M_\infty \leq \frac{a}{2})}{\mathbb{P}(M_\infty \leq a)} \inf_{y \in (0, y_{\min})} \frac{\mathbb{P}(M_\infty \leq a+y)}{f_M(a+y)} \inf_{y \in (0, y_{\min})} \frac{f_M\left(\frac{a+y-2y_{\min}}{2}\right)}{\mathbb{P}(M_\infty \leq \frac{a+y-2y_{\min}}{2})} \\
&\leq C \frac{\mathbb{P}(M_\infty \leq \frac{a}{2})}{\mathbb{P}(M_\infty \leq a)} \frac{\mathbb{P}(M_\infty \leq a+y_{\min})}{\mathbb{P}(M_\infty \leq \frac{a-2y_{\min}}{2})} \frac{\inf_{y \in (0, y_{\min})} f_M\left(\frac{a+y-2y_{\min}}{2}\right)}{\sup_{y \in (0, y_{\min})} f_M(a+y)} \\
&\leq C \frac{\mathbb{P}(M_\infty \leq \frac{a}{2})}{\mathbb{P}(M_\infty \leq a)} \frac{\mathbb{P}(M_\infty \leq a+y_{\min})}{\mathbb{P}(M_\infty \leq \frac{a-2y_{\min}}{2})} \frac{\int_0^{y_{\min}} f_M\left(\frac{a+y-2y_{\min}}{2}\right) dy}{\int_0^{y_{\min}} f_M(a+y) dy} \\
&= C \frac{\mathbb{P}(M_\infty \leq \frac{a}{2})}{\mathbb{P}(M_\infty \leq \frac{a-2y_{\min}}{2})} \frac{\mathbb{P}(M_\infty \leq a+y_{\min})}{\mathbb{P}(M_\infty \leq a)} \frac{\mathbb{P}\left(M_\infty \in \left(\frac{a-2y_{\min}}{2}, \frac{a-y_{\min}}{2}\right)\right)}{\mathbb{P}(M_\infty \in (a, a+y_{\min}))}
\end{aligned}$$

$$\begin{aligned}
&\lesssim C \frac{\mathbb{P}\left(B^* \in \left(\frac{a-2y_{\min}}{2}, \frac{a-y_{\min}}{2}\right)\right)}{\mathbb{P}(B^* \in (a, a+y_{\min}))} \\
&\lesssim C \frac{\mathbb{P}\left(B > \frac{a-2y_{\min}}{2}\right)}{\mathbb{P}(B > a+y_{\min})} \\
&\sim C \left(1 - \frac{3y_{\min}}{a+y_{\min}}\right)^{-\alpha}.
\end{aligned}$$

Here, the fourth inequality follows from Theorem 1. Substituting this into (7.10) gives

$$\begin{aligned}
\sup_{a \geq a_\rho^*} \mathbb{E}[\sigma(a) \mid \sigma(a) < \tau] &\lesssim \sup_{a \geq a_\rho^*} \frac{C}{(1-\rho)^2} \left(1 + \left(1 - \frac{2y_{\min}}{a+y_{\min}}\right)^{-\alpha}\right) \\
&\quad + \frac{C}{1-\rho} + \sup_{a \geq a_\rho^*} \frac{C}{1-\rho} a^{1-\alpha+\nu} + \sup_{a \geq a_\rho^*} \frac{C}{(1-\rho)a}.
\end{aligned}$$

Since all suprema are obtained in $a = a_\rho^*$ as $\rho \uparrow 1$, the above expressions can be written in terms of $1 - \rho$:

$$\begin{aligned}
\sup_{a \geq a_\rho^*} \mathbb{E}[\sigma(a) \mid \sigma(a) < \tau] &\lesssim \frac{C}{(1-\rho)^2} + \frac{C}{1-\rho} + C(1-\rho)^{\alpha-2-\nu} \left(\log \frac{1}{1-\rho}\right)^{1-\alpha+\nu} \\
&\quad + \frac{C}{\log \frac{1}{1-\rho}} \\
&= \mathcal{O}\left(\frac{1}{(1-\rho)^2}\right)
\end{aligned}$$

as $\rho \uparrow 1$.

8 Asymptotics of the first hitting time of level zero

This final section is devoted to the proof Theorem 5. We will validate expression (3.6);

$$\sup_{x \geq x_\rho^*} \left| \frac{\mathbb{P}(\tau > x; M_\tau > (1-\rho)x)}{\mathbb{P}(M_\tau > (1-\rho)x)} - 1 \right| \rightarrow 0 \quad (3.6, \text{ revisited})$$

as $\rho \uparrow 1$, as well as expression (3.8)

$$\sup_{x \geq x_\rho^*} \left| \frac{\mathbb{P}(\tau > x)}{\mathbb{P}(M_\tau > (1-\rho)x)} - 1 \right| \rightarrow 0 \quad (3.8, \text{ revisited})$$

as $\rho \uparrow 1$. Expressions (3.6) and (3.8) together imply (3.7) by an argument similar to (3.12).

Most of this section is dedicated to the proof of (3.8). Section 8.1 provides a lower bound for the probability $\mathbb{P}(\tau > x)$ by means of a strong approximation of the compound Poisson process $X(t)$. Section 8.2 considers an upper bound for the probability $\mathbb{P}(\tau > x)$ by means of a sample-path analysis that makes a distinction based on the supremum M_τ . The resulting events are again discriminated based on the number of jumps before τ or the first passage time of a specific level. The structure of the proof is visualized in Figure 2. The proof of expression (3.6) is contained in Section 8.1 after noting that $\frac{\mathbb{P}(\tau > x; M_\tau > (1-\rho)x)}{\mathbb{P}(M_\tau > (1-\rho)x)} \leq 1$.

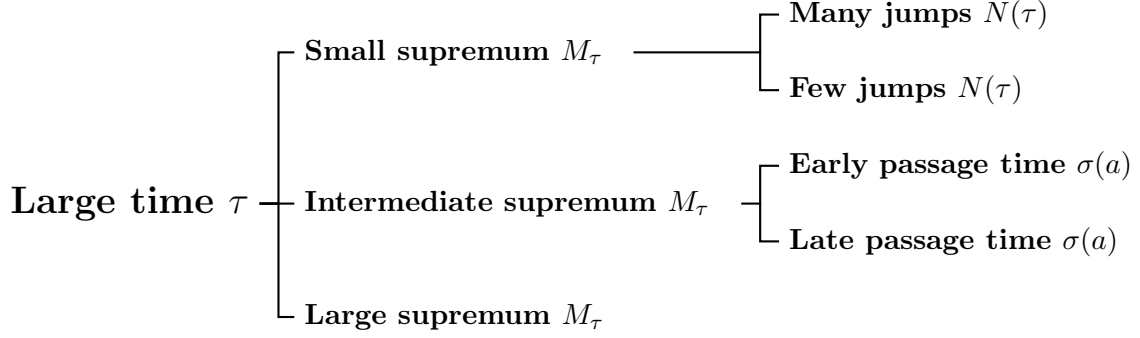


Figure 2: Visualization of proof structure. The event of a large time τ is analysed under three scenarios, depending on the size of the supremum M_τ . Two of these scenarios are again considered in more detail, where a distinction is based on the number of jumps before τ and the passage time of a high level a .

Sections 8.1 and 8.2.5 rely on the strong approximation of $X(t)$, as facilitated by Komlós et al. [23] and Major [27]. For a standard Brownian motion $\beta(t)$, their contributions imply

$$\begin{aligned}
X(t) &= B_0 + \sum_{i=1}^{N(t)} B_i - t \\
&= \sigma_B \left(\frac{1}{\sigma_B} \sum_{i=0}^{N(t)} B_i - \frac{N(t)+1}{\sigma_B} \mathbb{E}[B] \right) + (N(t)+1)\mathbb{E}[B] - t \\
&\rightarrow \sigma_B \beta(N(t)) + o\left(N(t)^{\frac{1}{2}}\right) + (N(t)+1)\mathbb{E}[B] - t
\end{aligned}$$

almost surely as $N(t) \rightarrow \infty$. As $N(t) \rightarrow \lambda t$ almost surely by the strong law of large numbers, it follows that

$$X(t) \rightarrow -(1-\rho)t + \sigma_B \beta(\lambda t) + o\left(t^{\frac{1}{2}}\right) \quad (8.1)$$

almost surely as $t \rightarrow \infty$. The strong approximation specifies the asymptotic behaviour of $X(t)$ as a Brownian motion with drift, subject to some noise. The well-studied behaviour of Brownian motions then allows us to analyse the behaviour of $X(t)$ on large time scales.

8.1 Lower bound

Fix $p \in (\frac{1}{2}, 1)$ and define

$$h_l(x, \rho) := (1-\rho)x + g(x, \rho), \quad (8.2)$$

where

$$g(x, \rho) := (1-\rho)^{2p-1} x^p. \quad (8.3)$$

Following Zwart [36] in his proof of Proposition 3.1, the first hitting time τ is lower bounded by

$$\begin{aligned}
\mathbb{P}(\tau > x) &\geq \mathbb{P}(\tau > x; M_\tau > (1-\rho)x) \\
&\geq \mathbb{P}(\tau > x; M_\tau > h_l(x, \rho)) \\
&\geq \mathbb{P}(X(t) > 0; 0 \leq t \leq x \mid X(0) = h_l(x, \rho)) \mathbb{P}(M_\tau > h_l(x, \rho)).
\end{aligned} \quad (8.4)$$

Substitution of (8.1) gives

$$\begin{aligned}\mathbb{P}(\tau > x; M_\tau > h_l(x, \rho)) &\geq \mathbb{P}\left(\inf_{t \in [0, x]} X(t) > -h_l(x, \rho)\right) \mathbb{P}(M_\tau > h_l(x, \rho)) \\ &= \mathbb{P}\left(\inf_{t \in [0, x]} -(1-\rho)t + \sigma_B \beta(\lambda t) + o\left(t^{\frac{1}{2}}\right) > -h_l(x, \rho)\right) \\ &\quad \times \mathbb{P}(M_\tau > h_l(x, \rho)).\end{aligned}$$

Using the facts $\inf_{s \in [0, x]} A(s) = -\sup_{s \in [0, x]} \{-A(s)\}$ and $\inf_{s \in [0, x]} \{A(s) + B(s)\} \geq \inf_{s \in [0, x]} A(s) + \inf_{s \in [0, x]} B(s)$ for all $A(s), B(s)$, we find that

$$\begin{aligned}\mathbb{P}(\tau > x; M_\tau > h_l(x, \rho)) &\geq \mathbb{P}\left(\sup_{t \in [0, x]} (1-\rho)t + \sigma_B \beta(\lambda t) < (1-\rho)x + (1+o(1))g(x, \rho)\right) \\ &\quad \times \mathbb{P}(M_\tau > h_l(x, \rho)).\end{aligned}$$

Scaling by a factor $1-\rho$ yields

$$\begin{aligned}\mathbb{P}(\tau > x; M_\tau > h_l(x, \rho)) &\geq \mathbb{P}\left(\sup_{s \in [0, (1-\rho)^2 x]} s + \sigma_B \beta(\lambda s) < (1-\rho)^2 x + (1+o(1))((1-\rho)^2 x)^p\right) \\ &\quad \times \mathbb{P}(M_\tau > (1-\rho)x) \times \frac{\mathbb{P}(M_\tau > h_l(x, \rho))}{\mathbb{P}(M_\tau > (1-\rho)x)}\end{aligned}\tag{8.5}$$

by standard scaling properties of Brownian motions. Applying Theorem 2 to the ratio in the latter expression gives

$$\begin{aligned}\sup_{x \geq x_\rho^*} \frac{\mathbb{P}(M_\tau > h_l(x, \rho))}{\mathbb{P}(M_\tau > (1-\rho)x)} &\sim \sup_{x \geq x_\rho^*} \frac{\mathbb{P}(B > h_l(x, \rho))}{\mathbb{P}(B > (1-\rho)x)} \\ &\sim (1 + ((1-\rho)^2 x_\rho^*)^{p-1})^{-\alpha},\end{aligned}\tag{8.6}$$

which tends to one as $\rho \uparrow 1$. The first term in (8.5) denotes the first hitting time distribution of a Brownian motion with unit positive drift and variance $\sigma_B^2 \lambda$. Let $\Phi(x)$ be the distribution function of a standard Gaussian and define $\bar{\Phi}(x) := 1 - \Phi(x)$. The following lemma provides an upper and lower bound for this first hitting time.

Lemma 2. *For any $t > 0, b > 0$ and $\bar{\sigma}_\xi > 0$ we have*

$$\mathbb{P}\left(\sup_{s \in [0, t]} s + \bar{\sigma}_\xi \beta(s) < b\right) > \Phi\left(\frac{b-t}{\bar{\sigma}_\xi \sqrt{t}}\right) - \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{b-t}{\bar{\sigma}_\xi \sqrt{t}}\right)^2\right] \frac{\bar{\sigma}_\xi \sqrt{t}}{b+t}.\tag{8.7}$$

Additionally, if $t - b > 0$ then

$$\mathbb{P}\left(\sup_{s \in [0, t]} s + \bar{\sigma}_\xi \beta(s) < b\right) < \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{t-b}{\bar{\sigma}_\xi \sqrt{t}}\right)^2\right] \frac{\bar{\sigma}_\xi \sqrt{t}}{t-b}.\tag{8.8}$$

Proof. From Corollary 1.8.7 in [20], we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{s \in [0, t]} s + \bar{\sigma}_\xi \beta(s) < b\right) &= \Phi\left(\frac{b-t}{\bar{\sigma}_\xi \sqrt{t}}\right) - \exp\left[\frac{2b}{\bar{\sigma}_\xi^2}\right] \Phi\left(-\frac{b+t}{\bar{\sigma}_\xi \sqrt{t}}\right) \\
&= \Phi\left(\frac{b-t}{\bar{\sigma}_\xi \sqrt{t}}\right) - \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{b-t}{\bar{\sigma}_\xi \sqrt{t}}\right)^2\right] \\
&\quad \times \sqrt{2\pi} \exp\left[\frac{1}{2}\left(\frac{b+t}{\bar{\sigma}_\xi \sqrt{t}}\right)^2\right] \bar{\Phi}\left(\frac{b+t}{\bar{\sigma}_\xi \sqrt{t}}\right). \tag{8.9}
\end{aligned}$$

According to equation (18.2.21) in Cuyt et al. [12], for any $y > 0$, Mills's ratio $m(y)$ for the standard normal distribution satisfies

$$m(y) := \frac{\bar{\Phi}(y)}{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}} < \frac{1}{y}. \tag{8.10}$$

Inequality (8.7) follows from applying this bound to (8.9). Inequality (8.8) is obtained by discarding the entire negative term of (8.9) and analyzing the first term with Mills's ratio. \square

Applying Lemma 2 to (8.5) yields

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s \in [0, (1-\rho)^2 x]} s + \sigma_B \beta(\lambda s) < (1-\rho)^2 x + (1+o(1))((1-\rho)^2 x)^p\right) \\
&> \Phi\left(\frac{(1+o(1))((1-\rho)^2 x)^{p-\frac{1}{2}}}{\sigma_B \sqrt{\lambda}}\right) \\
&\quad - \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(1+o(1))^2((1-\rho)^2 x)^{2p-1}}{2\sigma_B^2 \lambda}\right] \times \frac{\sigma_B \sqrt{\lambda}}{2((1-\rho)^2 x)^{\frac{1}{2}} + (1+o(1))((1-\rho)^2 x)^{p-\frac{1}{2}}}, \tag{8.11}
\end{aligned}$$

which tends to one as $(1-\rho)^2 x$ tends to infinity. Consequently, we conclude from (8.5), (8.6) and (8.11) that

$$\lim_{\rho \uparrow 1} \inf_{x \geq x_\rho^*} \frac{\mathbb{P}(\tau > x; M_\tau > h_l(x, \rho))}{\mathbb{P}(M_\tau > (1-\rho)x)} - 1 \geq 0 \tag{8.12}$$

and thus, by (8.4) and (8.12),

$$\lim_{\rho \uparrow 1} \inf_{x \geq x_\rho^*} \frac{\mathbb{P}(\tau > x; M_\tau > (1-\rho)x)}{\mathbb{P}(M_\tau > (1-\rho)x)} - 1 \geq 0 \tag{8.13}$$

and

$$\lim_{\rho \uparrow 1} \inf_{x \geq x_\rho^*} \frac{\mathbb{P}(\tau > x)}{\mathbb{P}(M_\tau > (1-\rho)x)} - 1 \geq 0. \tag{8.14}$$

Expression (8.13) completes the proof of (3.6). Furthermore, expression (8.14) concludes the lower bound of $\mathbb{P}(\tau > x)$.

8.2 Upper bound

The event $\{\tau > x\}$ is analysed by discriminating various scenarios, as visualized in Figure 2 at the beginning of this section. Firstly, we specify scenarios $\{\tau > x, M_\tau \in \cdot\}$, where the supremum M_τ can be in three regions: small, intermediate and large. The large M_τ region is related to

$\mathbb{P}(M_\tau > (1 - \rho)x)$, and is shown to be asymptotically equivalent to $\mathbb{P}(\tau > x)$. Secondly, the small and intermediate regions are analysed independently in Sections 8.2.1 and Sections 8.2.4.

Let $\delta_\rho = \left(\log \frac{1}{1-\rho}\right)^{-(1+\varepsilon_\delta)}$ for some $0 < \varepsilon_\delta < k - 1$. For any $p \in (\frac{1}{2}, 1)$ and $0 < \varepsilon_H < \frac{1}{2}(k - 1 - \varepsilon_\delta)\alpha$, define

$$h_u(x, \rho) := H_\rho(1 - \rho)x - g(x, \rho), \quad (8.15)$$

where

$$H_\rho := 1 - \left(\log \frac{1}{1-\rho}\right)^{-\varepsilon_H} \quad (8.16)$$

tends to one as $\rho \uparrow 1$, and

$$g(x, \rho) = (1 - \rho)^{2p-1}x^p \quad (8.17)$$

like before. Then

$$\begin{aligned} \mathbb{P}(\tau > x) &= \mathbb{P}(\tau > x; M_\tau > h_u(x, \rho)) \\ &\quad + \mathbb{P}(\tau > x; M_\tau \leq h_u(x, \rho)) \\ &\leq \mathbb{P}(M_\tau > h_u(x, \rho)) + \mathbb{P}(\tau > x; M_\tau \leq \delta_\rho(1 - \rho)x) \\ &\quad + \mathbb{P}(\tau > x; M_\tau \in (\delta_\rho(1 - \rho)x, h_u(x, \rho)]) \\ &= \mathbb{P}(M_\tau > h_u(x, \rho)) + I + II. \end{aligned}$$

Using Theorem 2, it follows that

$$\begin{aligned} \sup_{x \geq x_\rho^*} \frac{\mathbb{P}(M_\tau > h_u(x, \rho))}{\mathbb{P}(M_\tau > (1 - \rho)x)} &\lesssim \sup_{x \geq x_\rho^*} \frac{\mathbb{P}(B > h_u(x, \rho))}{\mathbb{P}(B > (1 - \rho)x)} \\ &\sim \sup_{x \geq x_\rho^*} \left(\frac{H_\rho(1 - \rho)x - (1 - \rho)^{2p-1}x^p}{(1 - \rho)x} \right)^{-\alpha} \\ &= \left(\frac{1}{H_\rho - \left(\log \frac{1}{1-\rho}\right)^{k\alpha(1-p)}} \right)^\alpha \\ &\rightarrow 1 \end{aligned}$$

as $\rho \uparrow 1$.

8.2.1 Small supremum M_τ : term I

Term I is the probability of a large first hitting time τ for which the corresponding process supremum M_τ is relatively small. First we show that the number of jumps before τ is not a lot higher than the expected number of jumps. Then, we show that it is highly unlikely to have a large τ that is caused by a probable amount of small jumps.

Recall that $N(t)$ denotes the number of jumps during an interval of length t . In particular, $N(t)$ is Poisson distributed with mean λt . Let $N_0(t)$ be the number of jumps of size at most $\delta_\rho(1 - \rho)x$ and $N_1(t)$ be the number of jumps of at least that size. Then $N_0(t)$ is Poisson distributed with mean $\lambda t \mathbb{P}(B < \delta_\rho(1 - \rho)x)$, $N_1(t)$ is Poisson distributed with mean $\lambda t \mathbb{P}(B \geq \delta_\rho(1 - \rho)x)$ and $N(t) \stackrel{d}{=} N_0(t) + N_1(t)$. Let \tilde{B}_i be i.i.d. random variables with distribution $\mathbb{P}(\tilde{B}_i \leq y) = \mathbb{P}(B \leq y \mid B \leq \delta_\rho(1 - \rho)x)$.

It is straightforward to see that if $\tau > x$, then all jumps before time x had a cumulative size of at least x ; that is, $\sum_{i=0}^{N(x)} \tilde{B}_i > x$ is a necessary condition for $\tau > x$. Furthermore, the

inequality $M_\tau \geq B_\tau$ is trivial. Let $\lambda^* = (1 + \eta_\rho)\lambda$, where $\eta_\rho = (1 - \rho)/2$. Note that this implies $\lambda^*\mathbb{E}[B] < 1$ provided that $\rho \in [0, 1)$. Now

$$\begin{aligned}
& \mathbb{P}(\tau > x, M_\tau \leq \delta_\rho(1 - \rho)x) \\
&= \mathbb{P}\left(\tau > x, \sum_{i=0}^{N(x)} B_i > x, M_\tau \leq \delta_\rho(1 - \rho)x\right) \\
&\leq \mathbb{P}\left(\sum_{i=0}^{N(x)} B_i > x, \bigvee_{i=0}^{N(x)} B_i \leq \delta_\rho(1 - \rho)x\right) \\
&= \mathbb{P}\left(\sum_{i=0}^{N_0(x)} B_{0,i} > x, N_1(x) = 0\right) \\
&\leq \mathbb{P}\left(\sum_{i=0}^{N_0(x)} B_{0,i} > x\right) \\
&\leq \mathbb{P}(N_0(x) \geq \lambda^*x) + \mathbb{P}\left(\sum_{i=0}^{\lambda^*x} B_{0,i} > x\right) \\
&\leq \mathbb{P}(N(x) \geq \lambda^*x) + \mathbb{P}\left(\sum_{i=0}^{\lambda^*x} B_i > x \mid \bigvee_{i=0}^{\lambda^*x} B_i \leq \delta_\rho(1 - \rho)x\right) \\
&= Ia + Ib.
\end{aligned}$$

Term Ia corresponds to a system where the number of jumps greatly exceeds its expectation. Term Ib indicates a likely number of jumps, none of which has a size exceeding $\delta_\rho(1 - \rho)x$.

8.2.2 Many jumps: term Ia

Term Ia is easily analysed by well-known results. From Markov's inequality, one can see that for all $s \geq 0$ we have

$$\begin{aligned}
Ia &= \mathbb{P}(e^{sN(x)} \geq e^{s\lambda^*x}) \\
&\leq e^{-s\lambda^*x} \mathbb{E}[e^{sN(x)}] \\
&= \exp[-\lambda x((1 + \eta_\rho)s - e^s + 1)].
\end{aligned}$$

Taking the infimum over all $s \geq 0$ gives

$$\begin{aligned}
Ia &\leq \exp[-\lambda x \sup_{s \geq 0} ((1 + \eta_\rho)s - e^s + 1)] \\
&= \exp[-\lambda x ((1 + \eta_\rho) \log(1 + \eta_\rho) - \eta_\rho)].
\end{aligned}$$

The bound $\log(1 + \eta_\rho) \geq \frac{2\eta_\rho}{2 + \eta_\rho}$ for $\eta_\rho > 0$ yields

$$Ia \leq \exp\left[-\frac{\eta_\rho^2}{2 + \eta_\rho} \lambda x\right].$$

Dividing by $\mathbb{P}(M_\tau > (1 - \rho)x)$, applying Theorem 2, taking the supremum and applying Potter's bound gives

$$\begin{aligned} \sup_{x \geq x_\rho^*} \frac{Ia}{\mathbb{P}(M_\tau > (1 - \rho)x)} &\lesssim \sup_{x \geq x_\rho^*} C \frac{1 - \rho}{\rho} \exp \left[(\alpha + \nu) \log \{(1 - \rho)x\} - \frac{\eta_\rho^2}{2 + \eta_\rho} \lambda x \right] \\ &\leq C \frac{1 - \rho}{\rho} \exp \left[(\alpha + \nu) \log \frac{1}{1 - \rho} - \frac{\lambda}{10} \log^{k\alpha} \frac{1}{1 - \rho} \right. \\ &\quad \left. + (\alpha + \nu) k \alpha \log \log \frac{1}{1 - \rho} \right] \\ &\rightarrow 0 \end{aligned} \tag{8.18}$$

as $\rho \uparrow 1$.

8.2.3 Few jumps: term Ib

Now consider term Ib . The corresponding event is a large τ , caused by not a reasonable amount of small jumps. The following Theorem by Prokhorov [32] is used to show that this scenario is unlikely as ρ tends to 1.

Theorem 6 (32, Theorem 1). *Suppose that $\xi_i, i = 1, \dots, n$ are independent, zero-mean random variables such that there exists a constant c for which $|\xi_i| \leq c$ for $i = 1, \dots, n$, and $\sum_{i=1}^n \text{Var}\{\xi_i\} < \infty$. Then*

$$\mathbb{P} \left(\sum_{i=1}^n \xi_i > x \right) \leq \exp \left\{ -\frac{x}{2c} \operatorname{arcsinh} \frac{xc}{2 \sum_{i=1}^n \text{Var}\{\xi_i\}} \right\}. \tag{8.19}$$

Using the bound $\operatorname{arcsinh}(z) = \log(z + \sqrt{1 + z^2}) \geq \log(2z)$, Prokhorov's inequality implies

$$\mathbb{P} \left(\sum_{i=1}^n \xi_i > x \right) \leq \left(\frac{cx}{\sum_{i=1}^n \text{Var}\{\xi_i\}} \right)^{-\frac{x}{2c}}. \tag{8.19*}$$

Define $Y_i := B_i - \mathbb{E}[B]$. Then

$$Ib \leq \mathbb{P} \left(\sum_{i=0}^{\lambda^* x} Y_i > (1 - \lambda^* \mathbb{E}[B])x \mid \bigvee_{i=1}^{\lambda^* x} Y_i \leq \delta_\rho (1 - \rho)x \right)$$

Let $\sigma_{B|B < a}^2$ be the variance of B provided $B < a$. Then $\sigma_{B|B < a}^2 \leq \sigma_B^2$. Using (8.19*), we obtain

$$\begin{aligned} Ib &\leq \left(\frac{1}{1 + \frac{1}{\lambda^* x}} \frac{\sigma_B^2}{\sigma_{B|B < a}^2} \frac{\delta_\rho (1 - \rho)x}{\sigma_B^2 \lambda^*} (1 - \lambda^* \mathbb{E}[B]) \right)^{-\frac{1 - \lambda^* \mathbb{E}[B]}{2\delta_\rho (1 - \rho)}} \\ &\lesssim \left(\frac{\delta_\rho (1 - \rho)x}{\sigma_B^2 \lambda^*} (1 - \lambda^* \mathbb{E}[B]) \right)^{-\frac{1 - \lambda^* \mathbb{E}[B]}{2\delta_\rho (1 - \rho)}}. \end{aligned}$$

Dividing by $\mathbb{P}(M_\tau > (1 - \rho)x) \sim \frac{\rho}{1 - \rho} \mathbb{P}(B > (1 - \rho)x)$ and applying Potter's bound yields

$$\begin{aligned} \frac{Ib}{\mathbb{P}(M_\tau > (1 - \rho)x)} &\lesssim C \frac{1 - \rho}{\rho} \exp \left[(\alpha + \nu) \log(1 - \rho)x \right. \\ &\quad \left. - \frac{1 - \lambda^* \mathbb{E}[B]}{2\delta_\rho (1 - \rho)} \log \left(\frac{\delta_\rho (1 - \rho)x}{\sigma_B^2 \lambda^*} (1 - \lambda^* \mathbb{E}[B]) \right) \right] \\ &= C \frac{1 - \rho}{\rho} \exp \left[\left(\alpha + \nu - \frac{1 - \lambda^* \mathbb{E}[B]}{2\delta_\rho (1 - \rho)} \right) \log(1 - \rho)x \right. \\ &\quad \left. - \frac{1 - \lambda^* \mathbb{E}[B]}{2\delta_\rho (1 - \rho)} \log \left(\frac{\delta_\rho}{\sigma_B^2 \lambda^*} (1 - \lambda^* \mathbb{E}[B]) \right) \right]. \end{aligned}$$

Substituting $\lambda^* = (1 + \eta_\rho)\lambda$ into the expression above, we find

$$\begin{aligned} \frac{Ib}{\mathbb{P}(M_\tau > (1 - \rho)x)} &\lesssim C \frac{1 - \rho}{\rho} \exp \left[\left(\alpha + \nu - \frac{1 - \frac{\rho}{1 - \rho} \eta_\rho}{2\delta_\rho} \right) \log(1 - \rho)x \right. \\ &\quad \left. - \frac{1 - \frac{\rho}{1 - \rho} \eta_\rho}{2\delta_\rho} \log \left\{ \frac{\delta_\rho \mathbb{E}[B]}{\rho \sigma_B^2} \left(\frac{1}{1 + \eta_\rho} - \rho \right) \right\} \right] \\ &= C \frac{1 - \rho}{\rho} \exp \left[\left(\alpha + \nu - \frac{1 - \frac{\rho}{1 - \rho} \eta_\rho}{2\delta_\rho} \right) \log(1 - \rho)x \right. \\ &\quad \left. - \frac{1 - \frac{\rho}{1 - \rho} \eta_\rho}{2\delta_\rho} \log \left\{ \frac{\delta_\rho \mathbb{E}[B]}{\rho \sigma_B^2} \left(\frac{1}{1 + \eta_\rho} - \rho \right) \right\} \right]. \end{aligned}$$

The supremum over $x \geq x_\rho^*$ is attained in $x = x_\rho^*$ for ρ sufficiently close to one. That is,

$$\begin{aligned} &\sup_{x \geq x_\rho^*} \frac{Ib}{\mathbb{P}(M_\tau > (1 - \rho)x)} \\ &\lesssim C \frac{1 - \rho}{\rho} \exp \left[\left(\alpha + \nu - \frac{1 - \frac{\rho}{1 - \rho} \eta_\rho}{2\delta_\rho} \right) \log(1 - \rho)x_\rho^* \right. \\ &\quad \left. - \frac{1 - \frac{\rho}{1 - \rho} \eta_\rho}{2\delta_\rho} \log \left\{ \frac{\delta_\rho \mathbb{E}[B]}{\rho \sigma_B^2} \left(\frac{1}{1 + \eta_\rho} - \rho \right) \right\} \right] \\ &= C \frac{1 - \rho}{\rho} \exp \left[\left(\alpha + \nu - \frac{2 - \rho}{4} \log^{1+\varepsilon_\delta} \frac{1}{1 - \rho} \right) \log \left\{ \frac{1}{1 - \rho} \log^{k\alpha} \frac{1}{1 - \rho} \right\} \right. \\ &\quad \left. - \frac{2 - \rho}{4} \log^{1+\varepsilon_\delta} \frac{1}{1 - \rho} \log \left\{ \frac{\delta_\rho \mathbb{E}[B]}{\rho \sigma_B^2} \left(\frac{2}{3 - \rho} - \rho \right) \right\} \right] \\ &= C \frac{1 - \rho}{\rho} \exp \left[\left(\alpha + \nu - \frac{2 - \rho}{4} \log^{1+\varepsilon_\delta} \frac{1}{1 - \rho} \right) \log \left\{ \frac{1}{1 - \rho} \log^{k\alpha} \frac{1}{1 - \rho} \right\} \right. \\ &\quad \left. - \frac{2 - \rho}{4} \log^{1+\varepsilon_\delta} \frac{1}{1 - \rho} \log \left\{ \frac{\delta_\rho \mathbb{E}[B]}{2\rho \sigma_B^2} \frac{(1 - \rho)(2 - \rho)}{3 - \rho} \right\} \right] \\ &= C \frac{1 - \rho}{\rho} \exp \left[(\alpha + \nu) \log \left\{ \frac{1}{1 - \rho} \log^{k\alpha} \frac{1}{1 - \rho} \right\} \right. \\ &\quad \left. - \frac{2 - \rho}{4} \log^{1+\varepsilon_\delta} \frac{1}{1 - \rho} \log \left\{ \frac{\mathbb{E}[B]}{2\rho \sigma_B^2} \frac{2 - \rho}{3 - \rho} \log^{k\alpha - 1 - \varepsilon_\delta} \frac{1}{1 - \rho} \right\} \right] \\ &\rightarrow 0 \end{aligned} \tag{8.20}$$

as $\rho \uparrow 1$. Together, (8.18) and (8.20) assure that term I is asymptotically dominated by $\mathbb{P}(M_\tau > (1 - \rho)x)$.

8.2.4 Intermediate supremum M_τ : term II

Recall that term II corresponds to the event of a large τ that experiences a intermediate supremum M_τ . More precisely, for $h_u(x, \rho) = H_\rho(1 - \rho)x - g(x, \rho)$, we defined

$$II = \mathbb{P}(\tau > x; M_\tau \in (\delta_\rho(1 - \rho)x, h_u(x, \rho)]), \tag{8.21}$$

so that

$$\begin{aligned}
\sup_{x \geq x_\rho^*} \frac{II}{\mathbb{P}(M_\tau > (1-\rho)x)} &\leq \sup_{x \geq x_\rho^*} \frac{\mathbb{P}(M_\tau > \delta_\rho(1-\rho)x)}{\mathbb{P}(M_\tau > (1-\rho)x)} \\
&\quad \times \sup_{x \geq x_\rho^*} \mathbb{P}(\tau > x; M_\tau \leq h_u(x, \rho) \mid M_\tau > \delta_\rho(1-\rho)x) \\
&\sim \delta_\rho^{-\alpha} \sup_{x \geq x_\rho^*} \mathbb{P}(\tau > x; M_\tau \leq h_u(x, \rho) \mid M_\tau > \delta_\rho(1-\rho)x). \tag{8.22}
\end{aligned}$$

By considering the time when the process $X(t)$ first exceeds level $\delta_\rho(1-\rho)x$, we can distinguish two rare events. Define the level $a_\rho := \delta_\rho(1-\rho)x$ and recall that the first passage time of this level is denoted by $\sigma(a_\rho)$. Set $\gamma_\rho := \left(\log \frac{1}{1-\rho}\right)^{-\varepsilon_\gamma}$ for some $\varepsilon_H < \varepsilon_\gamma < (k-1-\varepsilon_\delta)\alpha$ and note that $1-\gamma_\rho > H_\rho$. Now

$$\begin{aligned}
\mathbb{P}(\tau > x; M_\tau \leq h_u(x, \rho) \mid M_\tau > \delta_\rho(1-\rho)x) \\
&= \mathbb{P}(\tau > x; \sigma(a_\rho) \leq \gamma_\rho x; M_\tau \leq h_u(x, \rho) \mid \sigma(a_\rho) < \tau) \\
&\quad + \mathbb{P}(\tau > x; \sigma(a_\rho) > \gamma_\rho x; M_\tau \leq h_u(x, \rho) \mid \sigma(a_\rho) < \tau) \\
&\leq \mathbb{P}(\tau > x; \sigma(a_\rho) \leq \gamma_\rho x; M_\tau \leq h_u(x, \rho) \mid \sigma(a_\rho) < \tau) \\
&\quad + \mathbb{P}(\sigma(a_\rho) > \gamma_\rho x \mid \sigma(a_\rho) < \tau) \\
&= IIa + IIb.
\end{aligned}$$

Term IIa corresponds to a sample path where the process does not exceed level a before time $\gamma_\rho x$, provided that it will hit level a_ρ before it hits zero. Term IIb is associated with a sample path where the process reaches level a_ρ reasonably quick in almost the same setting; however, the process will not exceed level $h_u(x, \rho)$.

8.2.5 Early passage time: term IIa

Noting that $\tau > \tau - \sigma(a_\rho) > (1-\gamma_\rho)x$, term IIa is bounded by

$$\begin{aligned}
IIa &= \mathbb{P}(\tau > x; \sigma(a_\rho) < \gamma_\rho x; M_\tau \leq h_u(x, \rho) \mid \sigma(a_\rho) < \tau) \\
&\leq \mathbb{P}(\tau > (1-\gamma_\rho)x; M_\tau \leq h_u(x, \rho) \mid \sigma(a_\rho) < \tau).
\end{aligned}$$

As $X(\sigma(a_\rho)) \leq M_\tau$ when $\sigma(a_\rho) < \tau$, we have

$$IIa \leq \mathbb{P}(\tau > (1-\gamma_\rho)x; X(\sigma(a_\rho)) \leq h_u(x, \rho) \mid \sigma(a_\rho) < \tau).$$

Similar to the lower bound, we may use the strong approximation (8.1) to obtain

$$IIa \leq \mathbb{P}\left(\sup_{s \in [0, (1-\gamma_\rho)(1-\rho)^2x]} s + \sigma_B \beta(\lambda s) < H_\rho(1-\rho)^2x - (1+o(1))((1-\rho)^2x)^p\right).$$

Lemma 2 now implies

$$\begin{aligned}
IIa &< \frac{1}{\sqrt{2\pi}} \frac{\sigma_B \sqrt{\lambda(1-\gamma_\rho-H_\rho)}}{(1-\gamma_\rho-H_\rho)((1-\rho)^2x)^{\frac{1}{2}} + (1+o(1))((1-\rho)^2x)^{p-\frac{1}{2}}} \\
&\quad \times \exp\left[-\frac{1}{2}\left(\frac{(1-\gamma_\rho-H_\rho)((1-\rho)^2x)^{\frac{1}{2}} + (1+o(1))((1-\rho)^2x)^{p-\frac{1}{2}}}{\sigma_B \sqrt{\lambda(1-\gamma_\rho-H_\rho)}}\right)^2\right].
\end{aligned}$$

As the above expression decreases in x for all $x \geq x_\rho^*$, we obtain

$$\begin{aligned}
& \delta_\rho^{-\alpha} \sup_{x \geq x_\rho^*} IIa \\
& < \frac{1}{\sqrt{2\pi}} \frac{\sigma \sqrt{\lambda \left(\log \frac{1}{1-\rho} \right)^{-\varepsilon_H}}}{\left(1 - \left(\log \frac{1}{1-\rho} \right)^{\varepsilon_H - \varepsilon_\gamma} \right) \left(\log \frac{1}{1-\rho} \right)^{\frac{1}{2}(k-1-\varepsilon_\delta)\alpha - \varepsilon_H} + (1+o(1)) \left(\log \frac{1}{1-\rho} \right)^{(p-\frac{1}{2})(k-1-\varepsilon_\delta)\alpha}} \\
& \quad \times \exp \left[-\frac{1}{2} \left(\frac{\left(1 - \left(\log \frac{1}{1-\rho} \right)^{\varepsilon_H - \varepsilon_\gamma} \right) \left(\log \frac{1}{1-\rho} \right)^{\frac{1}{2}k\alpha - \varepsilon_H} + (1+o(1)) \left(\log \frac{1}{1-\rho} \right)^{(p-\frac{1}{2})k\alpha}}{\sigma \sqrt{\lambda \left(\log \frac{1}{1-\rho} \right)^{-\varepsilon_H}}} \right)^2 \right] \\
& \rightarrow 0
\end{aligned}$$

as $\rho \uparrow 1$.

8.2.6 Late passage time: Term IIb

Applying Markov's inequality and sequentially Theorem 4 to term IIb yields

$$\begin{aligned}
& \delta_\rho^{-\alpha} \sup_{x \geq x_\rho^*} IIb \leq \sup_{x \geq x_\rho^*} \frac{\mathbb{E}[\sigma(\delta_\rho(1-\rho)x) \mid \sigma(\delta_\rho(1-\rho)x) < \tau]}{\delta_\rho^\alpha \gamma_\rho x} \\
& = \mathcal{O} \left(\frac{1}{(1-\rho)^2} \right) \frac{(1-\rho)^2}{\left(\log \frac{1}{1-\rho} \right)^{(k-1-\varepsilon_\delta)\alpha - \varepsilon_\gamma}} \\
& \leq C \left(\frac{1}{\log \frac{1}{1-\rho}} \right)^{(k-1-\varepsilon_\delta)\alpha - \varepsilon_\gamma} \\
& \rightarrow 0
\end{aligned}$$

as $\rho \uparrow 1$.

This completes the proof of (3.8). The section is concluded by the proof of Corollary 1.

8.3 Number of jumps before the process hits level zero

Corollary 1 considers the number of jumps (including the initial jump) until $X(t)$ hits zero for the first time, $\tilde{N}(\tau) = N(\tau) + 1$, rather than the time at which the process hits zero. Let $A_i, i = 1, 2, \dots$ be exponentially distributed with rate $\rho/\mathbb{E}[B]$. Since $N(t)$ is Poisson distributed with the same rate, it follows that

$$N(t) \stackrel{d}{=} \begin{cases} 0 & \text{if } t < A_1 \\ \sup\{n \geq 1 : A_1 + \dots + A_{[n]} \leq t\} & \text{otherwise.} \end{cases} \quad (8.23)$$

Now, a key observation is that the process $X(t)$ has not hit zero after n jumps if and only if $X(t)$ has not hit zero after time $t = S_n^A := A_1 + \dots + A_{[n]}$; that is, $\{\tilde{N}(\tau) > n\} = \{\tau > S_n^A\}$. The corollary follows from this observation and a Chernoff bound.

Let $\chi_\rho = (1 - \rho)^{\varepsilon_\chi}$, where $0 < \varepsilon_\chi < 1$. Then

$$\begin{aligned}
\mathbb{P}\left(\tilde{N}(\tau) > n; \tau > \frac{n}{\lambda}\right) &= 1 - \mathbb{P}\left(\tilde{N}(\tau) \leq n; \tau \leq \frac{n}{\lambda}\right) \\
&\quad - \mathbb{P}\left(\tilde{N}(\tau) \leq n; \tau > \frac{n}{\lambda}\right) - \mathbb{P}\left(\tilde{N}(\tau) > n; \tau \leq \frac{n}{\lambda}\right) \\
&= 1 - \mathbb{P}\left(\tilde{N}(\tau) \leq n; \tau \leq \frac{n}{\lambda}\right) \\
&\quad - \mathbb{P}\left(\tilde{N}(\tau) \leq n; \tau > \frac{n}{\lambda}(1 + \chi_\rho)\right) - \mathbb{P}\left(\tilde{N}(\tau) \leq n; \tau \in \left(\frac{n}{\lambda}, \frac{n}{\lambda}(1 + \chi_\rho)\right]\right) \\
&\quad - \mathbb{P}\left(\tilde{N}(\tau) > n; \tau \leq \frac{n}{\lambda}(1 - \chi_\rho)\right) - \mathbb{P}\left(\tilde{N}(\tau) > n; \tau \in \left(\frac{n}{\lambda}(1 - \chi_\rho), \frac{n}{\lambda}\right]\right) \\
&\geq 1 - \mathbb{P}\left(\tau \leq \frac{n}{\lambda}\right) \\
&\quad - \mathbb{P}\left(S_n^A > \frac{n}{\lambda}(1 + \chi_\rho)\right) - \mathbb{P}\left(\tau \in \left(\frac{n}{\lambda}, \frac{n}{\lambda}(1 + \chi_\rho)\right]\right) \\
&\quad - \mathbb{P}\left(S_n^A < \frac{n}{\lambda}(1 - \chi_\rho)\right) - \mathbb{P}\left(\tau \in \left(\frac{n}{\lambda}(1 - \chi_\rho), \frac{n}{\lambda}\right]\right) \\
&= \mathbb{P}\left(\tau > \frac{n}{\lambda}\right) - \mathbb{P}\left(S_n^A > \frac{n}{\lambda}(1 + \chi_\rho)\right) - \mathbb{P}\left(S_n^A < \frac{n}{\lambda}(1 - \chi_\rho)\right) \\
&\quad - \mathbb{P}\left(\tau \in \left(\frac{n}{\lambda}(1 - \chi_\rho), \frac{n}{\lambda}(1 + \chi_\rho)\right]\right). \tag{8.24}
\end{aligned}$$

Both probabilities that consider the deviation of S_n^A from its mean can be bounded with a Chernoff argument. For any $t > 0$,

$$\begin{aligned}
\mathbb{P}\left(S_n^A > \frac{n}{\lambda}(1 + \chi_\rho)\right) &\leq \mathbb{E}\left[e^{tS_n^A}\right] \exp\left[-\frac{nt}{\lambda}(1 + \chi_\rho)\right] \\
&= \left(\frac{\lambda}{\lambda - t}\right)^n \exp\left[-\frac{nt}{\lambda}(1 + \chi_\rho)\right] \\
&= \exp\left[-n\left(\frac{t}{\lambda}(1 + \chi_\rho) - \log \frac{\lambda}{\lambda - t}\right)\right].
\end{aligned}$$

Taking the infimum over all $t > 0$ and using $\log(1 + x) \leq \frac{x}{2} \frac{2+x}{1+x} \leq x - \frac{x^2}{4}$ for $x \in (0, 1]$ yields

$$\begin{aligned}
\mathbb{P}\left(S_n^A > \frac{n}{\lambda}(1 + \chi_\rho)\right) &\leq \exp[-n(\chi_\rho - \log(1 + \chi_\rho))] \\
&\leq \exp\left[-n\frac{\chi_\rho^2}{4}\right]. \tag{8.25}
\end{aligned}$$

Similarly, using $\log(1 - x) \leq \frac{-2x}{2-x} \leq -x - \frac{x^2}{4}$ for $x \in (0, 1]$, one may show that

$$\mathbb{P}\left(S_n^A < \frac{n}{\lambda}(1 - \chi_\rho)\right) = \mathbb{P}\left(e^{-tS_n^A} > e^{-t\frac{n}{\lambda}(1 - \chi_\rho)}\right) \leq \exp\left[-n\frac{\chi_\rho^2}{4}\right]. \tag{8.26}$$

From (8.24), (8.25), (8.26) and Theorems 5 and 2 it follows that, for some non-increasing

function $\phi(x) \downarrow 0$,

$$\begin{aligned}
\sup_{n \geq x_\rho^*} 1 - \frac{\mathbb{P}(\tilde{N}(\tau) > n; \tau > \frac{n}{\lambda})}{\mathbb{P}(\tau > \frac{n}{\lambda})} &\leq 2 \sup_{n \geq x_\rho^*} \frac{1}{\mathbb{P}(\tau > \frac{n}{\lambda})} \exp \left[-n \frac{\chi_\rho^2}{4} \right] \\
&\quad + \sup_{n \geq x_\rho^*} \frac{\mathbb{P}(\tau > \frac{n}{\lambda}(1 - \chi_\rho))}{\mathbb{P}(\tau > \frac{n}{\lambda})} - \inf_{n \geq x_\rho^*} \frac{\mathbb{P}(\tau > \frac{n}{\lambda}(1 + \chi_\rho))}{\mathbb{P}(\tau > \frac{n}{\lambda})} \\
&\leq 2(1 + \phi(x_\rho^*)) \sup_{n \geq x_\rho^*} \frac{1 - \rho}{\rho} \left(\frac{(1 - \rho)n}{\lambda} \right)^{\alpha + \nu} \exp \left[-n \frac{\chi_\rho^2}{4} \right] \\
&\quad + \frac{1 + \phi(x_\rho^*)}{1 - \phi(x_\rho^*)} \sup_{n \geq x_\rho^*} \frac{\mathbb{P}(B > (1 - \rho)\frac{n}{\lambda}(1 - \chi_\rho))}{\mathbb{P}(B > (1 - \rho)\frac{n}{\lambda})} \\
&\quad - \frac{1 - \phi(x_\rho^*)}{1 + \phi(x_\rho^*)} \inf_{n \geq x_\rho^*} \frac{\mathbb{P}(B > (1 - \rho)\frac{n}{\lambda}(1 + \chi_\rho))}{\mathbb{P}(B > (1 - \rho)\frac{n}{\lambda})} \\
&\sim 2(1 + \phi(x_\rho^*)) \sup_{n \geq x_\rho^*} \exp \left[-n \frac{\chi_\rho^2}{4} + (\alpha + \nu) \log \frac{(1 - \rho)n}{\lambda} + \log(1 - \rho) \right] \\
&\quad + \frac{1 + \phi(x_\rho^*)}{1 - \phi(x_\rho^*)} (1 - \chi_\rho)^{-\alpha - \nu} - \frac{1 - \phi(x_\rho^*)}{1 + \phi(x_\rho^*)} (1 + \chi_\rho)^{-\alpha - \nu} \\
&\leq 2(1 + \phi(x_\rho^*)) \exp \left[-\frac{1}{4}(1 - \rho)^{-2(1 - \varepsilon_\chi)} \log^{k\alpha} \frac{1}{1 - \rho} + (\alpha + \nu + 1)(1 + o(1)) \log \frac{1}{1 - \rho} \right] \\
&\quad + \frac{1 + \phi(x_\rho^*)}{1 - \phi(x_\rho^*)} (1 - \chi_\rho)^{-\alpha - \nu} - \frac{1 - \phi(x_\rho^*)}{1 + \phi(x_\rho^*)} (1 + \chi_\rho)^{-\alpha - \nu} \\
&\rightarrow 0
\end{aligned}$$

as $\rho \uparrow 1$ by our choice of ε_χ . This proves (3.9).

The second part of the corollary follows if, additionally to (3.9), the expression

$$\sup_{n \geq x_\rho^*} \frac{\mathbb{P}(\tilde{N}(\tau) > n; \tau < \frac{n}{\lambda})}{\mathbb{P}(\tau > \frac{n}{\lambda})} \rightarrow 0 \tag{8.27}$$

holds as $\rho \uparrow 1$. Verification of this expression is a repetition of the above analysis, where we have shown

$$\begin{aligned}
\sup_{n \geq x_\rho^*} \frac{\mathbb{P}(\tilde{N}(\tau) > n; \tau < \frac{n}{\lambda})}{\mathbb{P}(\tau > \frac{n}{\lambda})} &\leq \sup_{n \geq x_\rho^*} \frac{\mathbb{P}(S_n^A < \frac{n}{\lambda}(1 - \chi_\rho))}{\mathbb{P}(\tau > \frac{n}{\lambda})} + \sup_{n \geq x_\rho^*} \frac{\mathbb{P}(\tau \in (\frac{n}{\lambda}(1 - \chi_\rho), \frac{n}{\lambda}])}{\mathbb{P}(\tau > \frac{n}{\lambda})} \\
&\leq (1 + \phi(x_\rho^*)) \exp \left[-\frac{1}{4}(1 - \rho)^{-2(1 - \varepsilon_\chi)} \log^{k\alpha} \frac{1}{1 - \rho} + (\alpha + \nu + 1)(1 + o(1)) \log \frac{1}{1 - \rho} \right] \\
&\quad + \frac{1 + \phi(x_\rho^*)}{1 - \phi(x_\rho^*)} (1 - \chi_\rho)^{-\alpha - \nu} - 1 \\
&\rightarrow 0.
\end{aligned}$$

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